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WEAK EXPECTATIONS AND INJECTIVITY IN OPERATOR ALGEBRAS

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ABSTRACT. An example is given of a noninjective von Neumann algebra M on a separable Hilbert space $\mathcal{H}$ for which there exists a weak expectation of $\mathcal{L}(\mathcal{H})$ into M. Some positive results about weak expectations are also obtained.

1. Introduction. Much of the recent work in the structure theory of C^*-algebras and von Neumann algebras has been concerned with the closely related concepts of nuclearity and injectivity (see [3], [4]). It is becoming evident that the classes of nuclear C^*-algebras and injective von Neumann algebras have a simple enough structure to be tractable (at least in the separable case), while being broad enough to include most algebras of interest.

Lance [7] introduced the concept of a weak expectation, and showed its close connection with the problem of extending cross norms on tensor products of C^*-algebras. Subsequently, Choi and Effros [2] made a more detailed study of weak expectations and injectivity. Lance raised a number of important questions, one of which (slightly rephrased) is the following:

Question. Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$. If there is a weak expectation for M, is M necessarily injective?

Choi and Effros [2], using a result of Wassermann [11], gave a negative answer if $\mathcal{H}$ is not separable. But in view of results such as those of [4], [5], and §2 of this paper, it is reasonable to expect the situation for a separable $\mathcal{H}$ to be nicer.

Nonetheless, the answer to Lance's question is negative, even if $\mathcal{H}$ is separable; a counterexample is given in §3. §4 is a discussion of open questions and applications.

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2. Positive results. In this section, we describe some of the structure associated with weak expectations, particularly when the underlying Hilbert space is separable.

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Throughout this section, let A be a C^*-algebra of operators on a Hilbert space $\mathcal{H}$, containing the identity, and let $M = A''$ be its weak closure. Recall that a weak expectation for M with respect to A is a completely positive map P from $\mathcal{L}(\mathcal{H})$ into M such that $P(1) = 1$ and $P(ax) = aP(x)$ and $P(xa) = P(x)a$ for all $a \in A$, $x \in \mathcal{L}(\mathcal{H})$. If $A = M$, P is called a conditional expectation, and M is said to be injective if there is a conditional expectation of $\mathcal{L}(\mathcal{H})$ onto M. Details can be found in **[2]**, [7], and **[10]**.

Unlike conditional expectations, weak expectations need not be idempotent; however, the next theorem shows that idempotent weak expectations can always be found.

THEOREM 2.1. *If there is a weak expectation for M with respect to A, there is an idempotent weak expectation for M with respect to A.*

PROOF. Let $\mathcal{S}$ be the set of all weak expectations for M with respect to A. $\mathcal{S}$ is a compact convex subset of $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ with the topology of pointwise σ-weak convergence. Let $P_0 \in \mathcal{S}$; then the function $P \mapsto P \circ P_0$ is a continuous map of $\mathcal{S}$ into $\mathcal{S}$, and so has a fixed point P_1 by the Schauder fixed point theorem. Similarly, there is a $P_2 \in \mathcal{S}$ such that $P_2 \circ P_1 = P_2$. Then $P_2 \circ P_0 = (P_2 \circ P_1) \circ P_0 = P_2 \circ (P_1 \circ P_0) = P_2 \circ P_1 = P_2$. Continuing inductively, we construct a sequence $\{P_n\}$ such that $P_n \circ P_m = P_n$ if $n > m$. Let P_ω be any limit point of the sequence $\{P_n\}$; then $P_\omega \circ P_n = P_\omega$ for all n. Thus the construction can be continued transfinitely to give a $P_\alpha \in \mathcal{S}$ for each ordinal α, such that $P_\alpha \circ P_\beta = P_\alpha$ if $\alpha > \beta$. Eventually, the transfinite sequence must repeat.

For the rest of the section, we will assume that P is an idempotent weak expectation for M with respect to A. We may assume that $A = \{a \in M: P(ax) = aP(x),\ P(xa) = P(x)a\ \forall x \in \mathcal{L}(\mathcal{H})\}$. Let R be the range of P. Then $A \subseteq R \subseteq M$, and by [**1**, Theorem 3.1], $A = \{a \in R: a^*a \in R, aa^* \in R\}$. Thus, A contains all projections of R, and if R is a Jordan algebra, then $A = R$ (so R is an algebra). By [**2**, Theorem 3.1] and [**10**, Theorem 5], R becomes a conditionally complete injective AW^*-algebra under the multiplication $x \cdot y = P(xy)$.

The next proposition shows that if $\mathcal{H}$ is separable, then "most" spectral projections of selfadjoint elements of A are again in A.

PROPOSITION 2.2. *Suppose $\mathcal{H}$ is separable. Let $x \in A$, $x = x^*$. Then x is in the C^*-subalgebra of A generated by the spectral projections of x which are in A.*

PROOF. Let $\lambda, \mu \in \mathbf{R}$, $\lambda < \mu$, λ and μ not in the point spectrum of x. Then $E_{[\lambda,\mu]}(x) = E_{(\lambda,\mu)}(x)$ is a projection of $\mathcal{L}(\mathcal{H})$ which is both an increasing and a decreasing limit of elements of A^+. Since P is order-preserving, $E_{[\lambda,\mu]}(x) \in R$. But every projection in R is in A. Since the point spectrum of x is countable, its complement is dense in $\mathbf{R}$.

COROLLARY 2.3. *If $\mathcal{H}$ is separable, every maximal commutative subalgebra of*

A is generated by projections, and the projections of A are strongly dense in the projections of M.

PROOF. Let e be a projection of M, and let $\{x_n\}$ be a sequence of selfadjoint elements of the unit ball of A with $x_n \to e$ strongly (Kaplansky density.) Let $\lambda_n \in [1/3, 2/3] \sim \sigma_p(x_n)$, and let $e_n = E_{[\lambda_n, 1]}(x_n)$. Then e_n is a projection in A, and $e_n \to e$ strongly.

REMARK. A similar proof shows that if A is a conditionally complete C^*-algebra which has a faithful representation π on a separable Hilbert space, then every maximal commutative C^*-subalgebra of $\pi(A)$ is generated by projections, and therefore A is an AW^*-algebra.

If $\mathcal{H}$ is separable and R is an algebra, in many cases we can conclude from [**5**] that the identity representation of R is normal, i.e. the supremum in R of an increasing sequence of projections of R is the same as the supremum in $\mathcal{L}(\mathcal{H})$. In particular, if M is a factor, then R is an AW^*-factor, so [**5**, Theorems 1 and 2] apply. The following proposition then shows that $R = M$.

PROPOSITION 2.4. *Let B be an AW^*-algebra, and let π be a faithful normal representation of B. Then B is a W^*-algebra, and $\pi(B)$ is weakly closed.*

PROOF. B is a W^*-algebra by [**8**, p. 174]. Let C be a maximal commutative subalgebra of B; C is a commutative W^*-algebra, and $\pi(C)$ is weakly closed by [**9**, 1.13.2 and 1.16.2]. Thus $\pi(B)$ is weakly closed by [**8**, p. 174].

We summarize the above results in a theorem.

THEOREM 2.5. *Let M be a factor on a separable Hilbert space. If there is an indempotent weak expectation P for M (with respect to some weakly dense C^*-subalgebra) whose range is a Jordan algebra, then P is a conditional expectation onto M, and so M is injective.*

The following argument, due to E. Effros, indicates that the hypothesis that $\mathcal{H}$ be separable is probably necessary. Let A be a hyperfinite II_1 factor; then A^{**} is not injective ([**11**, Corollary 1.9] and [**3**, Theorem 3]), so there ought to be a direct summand M of A^{**} which is a noninjective factor (if A were separable, this would follow from [**4**, 6.4(b) and 6.5]; the difficulty in the present case is the lack of a direct integral theory). If M is such a factor, let π be a representation of A on a Hilbert space $\mathcal{H}$ with $\pi(A)'' = M$. π is faithful since A is simple; since A is injective, there is a conditional expectation of $\mathcal{L}(\mathcal{H})$ onto $\pi(A)$, which is a weak expectation for M.

The hypothesis that M be a factor is also necessary: there exists a faithful representation π of l^∞ on a separable Hilbert space $\mathcal{H}$ such that $\pi(l^\infty)$ is not weakly closed (W. Bade, unpublished). Since l^∞ is injective, there is a conditional expectation of $\mathcal{L}(\mathcal{H})$ onto $\pi(l^\infty)$ which is a weak expectation for $M = \pi(l^\infty)''$.

3. Negative results. In this section we give an example of a noninjective von Neumann algebra M on a separable Hilbert space which has a weak expectation with respect to a weakly dense C^*-subalgebra.

Let $A = B = C^*(\mathbf{F}_2)$, the group C^*-algebra of the free group on two generators. Let $\rho_1, \rho_2, \ldots$ (respectively $\sigma_1, \sigma_2, \ldots$) be pairwise nonequivalent irreducible representations of A (resp. B) such that $\bigoplus \rho_i$ (resp. $\bigoplus \sigma_i$) is faithful. Let $\pi_i = \rho_i \otimes \sigma_i$ be the corresponding irreducible representation of $A \otimes B$ on $\mathcal{H}_i$ (throughout this section, "tensor product" will always mena "minimal tensor product", using the least C^*-cross norm). Let π_0 be the representation of $A \otimes B$ generated by the left and right regular representations of $\mathbf{F}_2$. (By [**11**, Proposition 2.7] the regular representation actually gives a representation of the minimal tensor product.) Let π_0 act on $\mathcal{H}_0$, and let $M_i = \pi_i(A \otimes 1)''$ for each $i \geqslant 0$. Let $\mathcal{H} = \bigoplus_{i=0}^{\infty}\mathcal{H}_i$, $M = \bigoplus_{i=0}^{\infty} M_i$, $\pi = \bigoplus_{i=0}^{\infty}\pi_i$. Then π is a faithful representation of $A \otimes B$, and $\pi(A \otimes 1)'' = \pi(1 \otimes B)' = M$ since all of the ρ_i's and σ_i's are nonequivalent. As in the proof of [**7**, Theorem 3.3], we may regard $A \otimes B$ as a subalgebra of $\mathcal{L}(\mathcal{H}) \otimes B$, and so π extends to a representation $\hat{\pi}$ of $\mathcal{L}(\mathcal{H}) \otimes B$ on a larger Hilbert space $\hat{\mathcal{H}}$. For $x \in \mathcal{L}(\mathcal{H})$, let $P(x)$ be the compression of $\hat{\pi}(x \otimes 1)$ to $\mathcal{H}$; then P is a weak expectation of $\mathcal{L}(\mathcal{H})$ into M with respect to $\pi(A \otimes 1)$. However, it is well known that M_0 is not injective, so M is not injective.

4. Open questions. An important feature of the weak expectation constructed in §3 is that the weak expectation algebra for P does not contain the center of M. The fact that $\pi(A \otimes 1)$ is skewed with respect to the center of M seems essential for the construction. In fact, Lance [**7**, Theorem 4.2] has shown that there is no weak expectation for M_0 with respect to $\pi_0(A \otimes 1)$.

It is natural to rephrase the original question in the following form:

Question. Let M be a von Neumann algebra on a separable Hilbert space. If there is a weak expectation for M with respect to a weakly dense subalgebra containing the center of M, is M necessarily injective?

Using direct integral theory, it is enough to settle the question for M a factor.

If the answer to this question is yes, then a modification of the techniques of [**7**] can be used to prove the following two consequences, providing a solution to two of the most important open questions concerning nuclear C^*-algebras:

(1) Every separable C^*-algebra can be embedded into a separable nuclear C^*-algebra.

(2) There exists a separable, simple, purely infinite, nuclear C^*-algebra with identity; such a C^*-algebra cannot be an inductive limit of type I C^*-algebras by [**6**].

Added in Proof. The above question and consequence (1) have been settled negatively by the author [**12**], and consequence (2) settled affirmatively by J. Cuntz [**13**].

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