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NORM CONDITIONS ON RESOLVENTS OF SIMILARITIES OF HILBERT SPACE OPERATORS AND APPLICATIONS TO DIRECT SUMS AND INTEGRALS OF OPERATORS

FRANK GILFEATHER[1]

ABSTRACT. Similarities of an operator T are determined so that on certain sets the norm of the resolvents of the similarity satisfy bounding conditions independent of T. The results are applied to show that direct sums and integrals of operators are quasi-similar to operators with spectrum depending only on the spectrum of the summands.

In this paper all operators will be bounded linear operators on Hilbert space. Let T be an operator and A be a compact subset of the plane disjoint from $\sigma(T)$, the spectrum of T. For any similarity S of T there are constants M_1 and M_2 so that if f is a function analytic on a neighborhood of $\sigma(T)$, then $\|f(S)\| \leqslant M_1\|f(T)\|$ and $\|(S - \lambda I)^{-1}\| \leqslant M_2$ for all $\lambda \in A$. The main result in this paper is to obtain similarities S of T so that the constants M_1 and M_2 are reasonably good. The best possible M_2 would be $M_0 = \operatorname{dist}(A, \sigma(T))^{-1}$, and, in fact, we show that any $M_2 > M_0$ is obtainable simultaneous with $M_1 = 1$.

These resolvent growth results are used to show that direct sums and integrals of operators are quasi-similar to operators with the smallest possible spectrum. In particular, if $T = \Sigma \oplus T_i$, then it is always the case that $\overline{\bigcup \sigma(T_i)} \subset \sigma(T)$. However, we show that there is a quasi-similarity $S = \Sigma \oplus S_i$ of T, with S_i similar to T_i for each i and $\sigma(S) = \overline{\bigcup \sigma(S_i)} = \overline{\bigcup \sigma(T_i)}$. We wish to thank Larry Fiakalow for several helpful conversations and bringing our attention to [2].

Whenever $\sigma(T)$ is contained in the interior of a disk of radius r, then there exists a similarity S of T so that $\|(S - \lambda I)^{-1}\| \leqslant (|\lambda| - r)^{-1}$ whenever $|\lambda| > r$. This follows from the well-known fact that the infimum of the norms of similarities of T is just the spectral radius of T [3]. For $|\lambda| > r$ the norms of $(S - \lambda I)^{-1}$ clearly do not depend on T other than on the spectral radius of T. The following two lemmas generalize this situation.

LEMMA 1. *Let T be an operator with $\sigma(T)$ contained in the open unit disk. There exists a similarity S of T so that $\|S\| \leqslant 1$ and $\|f(S)\| \leqslant \|f(T)\|$ for all functions analytic on a neighborhood of $\sigma(T)$.*

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PROOF. Define $R = (\Sigma_0^\infty T^{*n}T^n)^{1/2}$ and let $x \in H$. Then $\|Rx\|^2 = \langle R^2x, x\rangle = \langle \Sigma_0^\infty T^{*n}T^n x, x\rangle$ so that

$$\|Rx\|^2 = \sum_0^\infty \|T^n x\|^2. \tag{1}$$

Hence R is bounded above and below. Let $S = RTR^{-1}$; then by (1)

$$\begin{aligned}\|RTR^{-1}x\|^2 &= \sum_0^\infty \|T^{n+1}R^{-1}x\|^2\\ &= \sum_0^\infty \|T^nR^{-1}x\|^2 - \|R^{-1}x\|^2\\ &= \|x\|^2 - \|R^{-1}x\|^2.\end{aligned}$$

Therefore $\|S\| \leqslant 1$.

Now let f be a function analytic on a neighborhood of $\sigma(T)$. It follows again from (1) that

$$\begin{aligned}\|f(S)x\|^2 = \|Rf(T)R^{-1}x\|^2 &= \sum_0^\infty \|T^nf(T)R^{-1}x\|^2\\ &\leqslant \|f(T)\|^2 \sum_0^\infty \|T^nR^{-1}x\|^2.\end{aligned}$$

Hence, since $\Sigma_0^\infty \|T^nR^{-1}x\|^2 = \|x\|^2$, we obtain

$$\|f(S)\| \leqslant \|f(T)\|. \tag{2}$$

REMARK. In particular, $\|(S - \lambda I)^{-1}\| \leqslant \|(T - \lambda I)^{-1}\|$ for λ not in $\sigma(T)$. By their definitions it follows that the operators S and R are in the C^*-algebra generated by T and I.

If one simply manipulates an operator by scale, scalar change and inversion, then the preceding lemma can be used to get a bound on the resolvent of a similarity of T near a point in the resolvent of T.

LEMMA 2. *Assume α is not in $\sigma(T)$ and fix an $r < d = \operatorname{dist}(\alpha, \sigma(T))$. There exists a similarity S of T so that $\|(S - \lambda I)^{-1}\| \leqslant (r - |\lambda - \alpha|)^{-1}$ for λ such that $|\lambda - \alpha| < r$ and for which $\|f(S)\| \leqslant \|f(T)\|$ for functions analytic on a neighborhood of $\sigma(T)$.*

PROOF. Take $T_0 = r(T - I)^{-1}$ to be the operator T in Lemma 1. The operator T_0 satisfies the hypothesis of Lemma 1 so there exists an operator S_0 similar to T_0 such that $\|S_0\| \leqslant 1$ and $\|f(S_0)\| \leqslant \|f(T_0)\|$, whenever f is analytic on a domain containing $\sigma(T_0)$.

If we let $S = rS_0^{-1} + \alpha I$, then S is similar to T. Let f be analytic on $\sigma(T)$ and $\psi(z) = (r + \alpha z)/z$. If $g = f \circ \psi$, then g is analytic on $\sigma(S_0) = \sigma(T_0)$ so $\|g(S_0)\| \leqslant \|g(T_0)\|$. However $g(S_0) = f(S)$ and $g(T_0) = f(T)$ so $\|f(S)\| \leqslant \|f(T)\|$.

Finally the resolvent norm condition about α follows since $\|S_0\| \leqslant 1$. Specifically, $\|(S_0 - \gamma I)^{-1}\| \leqslant (|\gamma| - 1)^{-1}$, whenever $|\gamma| > 1$, thus

$$\left\|\left(r(S-\alpha I)^{-1}-\gamma I\right)^{-1}\right\| \leqslant (|\gamma|-1)^{-1}, \quad \text{for } |\gamma|>1.$$

This yields $\|(S-\alpha I)((1+\mu\alpha)I-\mu S)^{-1}\| \leqslant r(|\mu|r-1)^{-1}$, for $|\mu|>r^{-1}$. Hence

$$\left\|\left(\mu^{-1}(1+\mu\alpha)I-S\right)^{-1}\right\| \leqslant \left\|(S-\alpha)^{-1}\right\| \, |\mu| r(|\mu|r-1)^{-1},$$

for $|\mu|>r^{-1}$. However, $r(S-\alpha)^{-1}=S_0$ and $\|S_0\| \leqslant 1$, so letting $\delta = \mu^{-1}(1+\mu\alpha)$ we obtain $\|(\delta I-S)^{-1}\| \leqslant (r-|\delta-\alpha|)^{-1}$, for $|\delta-\alpha|<r$.

Just as in Lemma 1, the operators in this lemma all belong to the C^*-algebra generated by T. By repeated application of Lemma 2 we obtain the following theorem. We use D^0 to be the interior of a set and $\setminus D$ to be the complement of D.

THEOREM. *Let K, D be compact subsets of the plane so that $K \subset D^0$. For any constant $M > (\operatorname{dist}(K, \setminus D^0))^{-1}$ and for every operator T with $\sigma(T) \subset K$, there exists a similarity S of T so that for all functions f analytic on $\sigma(T)$,*

(1) $\|f(S)\| \leqslant \|f(T)\|$,

(2) $\|(S-\lambda I)^{-1}\| \leqslant M$ *for* $\lambda \notin D$,

(3) $\|S\| \leqslant \sup_{\lambda \in D}|\lambda|$.

PROOF. Let T be any operator with $\sigma(T) \subset K$. Let $d=\sup_{\lambda\in K}|\lambda|$ and $\varepsilon = \operatorname{dist}(K, \setminus D^0)$. Choose η so that $\varepsilon > \eta > 0$ and $(\varepsilon-\eta)^{-1} < M$. Then by Lemma 1, there exists a similarity S of T satisfying the norm condition (1) for all possible f where $\|S\| \leqslant d+\eta$. Thus we may assume T has norm less than $d+\eta$. From here on the proof is somewhat messy, but obvious. For $\lambda \notin D$, let $r(\lambda) = \operatorname{dist}(\lambda, K)$. Let $B_{t(\lambda)}$ denote the ball about λ of radius $t(\lambda)$ and B the disk of radius $d+\varepsilon$. Clearly for our result we may assume that $B \supset D$. Let $\eta(\lambda) = r(\lambda) - \varepsilon + \eta/2$; then since each $r(\lambda) \geqslant \varepsilon$ it follows that $\eta(\lambda) \geqslant \eta/2$. Consider the collection of sets $\{B_{\eta(\lambda)}\}$, where $\lambda \varepsilon \setminus D$.

By the compactness of $B \setminus D^0$, there are points $\lambda_1, \ldots, \lambda_k$ for which $\{B_{\eta(\lambda_i)}\}$ covers $B \setminus D^0$. Notice that ε and the λ's depend only on K and D (η depends on our choice of M).

If we apply Lemma 2 to λ_1 and T, letting the r in Lemma 2 be $r(\lambda_1) - \eta/2$, we obtain a similarity S_1 of T with nice resolvent norm properties. Specifically if $\lambda \in B_{\eta(\lambda_1)}$, then

$$\text{(1)} \qquad \left\|(S_1-\lambda I)^{-1}\right\| \leqslant \left(r(\lambda_1)-\eta/2-\eta(\lambda_1)\right)^{-1} \leqslant (\varepsilon-\eta)^{-1} \leqslant M,$$

and

$$\text{(2)} \qquad \|f(S_1)\| \leqslant \|f(T)\| \quad \text{for all possible } f.$$

If we apply the above to λ_2 and S_1, we obtain an S_2 satisfying (1) on the set $B_{\eta(\lambda_1)} \cup B_{\eta(\lambda_2)}$ and still satisfying $\|f(S_2)\| \leqslant \|f(T)\|$ for all possible f. After k-steps we obtain S_k which satisfies

(1) $\|(S_i-\lambda I)^{-1}\| \leqslant M$ on $B \setminus D^0$, and

(2) $\|f(S_k)\| \leqslant \|f(T)\|$ for all possible f. To see that $\|(S_k-\lambda I)^{-1}\| \leqslant M$ for $\lambda \notin B$ recall that $\|S_k\| \leqslant \|T\| \leqslant d+\eta$ and, thus,

$$\|(S_k - \lambda I)^{-1}\| \leqslant \operatorname{dist}(|\lambda|, \|S_k\|)^{-1} \leqslant (\varepsilon - \eta)^{-1} \leqslant M.$$

Thus we have S_k similar to T with properties of the theorem satisfied.

A corollary of the theorem says that once you have obtained a bound on the complement of one compact set, going to a smaller set does not disturb that bound. This fact is a consequence of property (1) in the theorem.

COROLLARY. *Let $\sigma(T) \subset K \subset D_1^0 \subset D_2^0$ where $D_1 \subset D_2$. If M_i are constants given in the previous theorem for D_i and K, then there exists a similarity S of T so that $\|(S - \lambda I)^{-1}\| \leqslant M_i$ if λ is not in D_i and S satisfies the other conditions of the theorem.*

The corollary can be used to obtain a result which was announced and independently obtained by D. Herrero [**2**].

PROPOSITION. *If $T = \sum \oplus T_n$ is a direct sum of operators, then T is quasi-similar to an operator S for which $\sigma(S) = \overline{\bigcup \sigma(T_n)}$.*

PROOF. Let $K = \overline{\bigcup \sigma(T_n)}$ and $D_n = K + 1/n$. Then there exists a similarity S_n of T_n so that $\|(S_n - \lambda)^{-1}\| \leqslant M_i$ if $\lambda \notin D_i$ for $i = 1, \ldots, n$. Furthermore we may assume that if $d = \sup_{\lambda \in K}|\lambda|$, then $\|S_n\| < d + 1$. Let $S = \sum \oplus S_n$ and suppose $T_n = R_n S_n R_n^{-1}$. It easily follows that if $X = \sum \oplus R_n/\|R_n\|$ and $Y = \sum \oplus R_n^{-1}/\|R_n^{-1}\|$, then $XT = SX$ and $TY = YS$. Finally, by the growth conditions on the resolvents of S_n, it follows that for $\lambda \notin K$, $\|(S_n - \lambda)^{-1}\|$ is uniformly bounded. Therefore, $\lambda \notin \sigma(S)$ and $\sigma(S) \subset K$.

The second application of the theorem involves direct integrals. For the details of direct integral decompositions, we refer to [**4**]; however, we shall introduce some basic notations and results here. Let μ be a finite positive regular measure defined on the Borel sets of a separable metric space Λ, and let e_n, $1 \leqslant n \leqslant \infty$, be a collection of disjoint Borel sets of Λ with union Λ. Let $H_1 \subset H_2 \subset \cdots \subset H_\infty$ be a sequence of Hilbert spaces, with H_n having dimension n and H_∞ being separable. By

$$H = \int_\Lambda^\oplus H(t)\,\mu(dt)$$

we shall denote the space of weakly μ-measurable functions from Λ into H_∞ such that $f(t) \in H_n$, if $t \in e_n$, and $\int_\Lambda \|f(t)\|^2 \mu(dt) < \infty$. The space H is a Hilbert space, and we shall denote the element $f \in H$ determined by the vector valued function $f(t)$ as $\int_\Lambda^\oplus f(t)\,\mu(dt)$.

An operator T on H is said to be *decomposable* if there exists a μ-measurable operator valued function $T(t)$ so that $(Tf)(t) = T(t)f(t)$ for $f \in H$. The operator T is denoted by

$$T = \int_\Lambda^\oplus T(t)\,\mu(dt).$$

It is easy to see that $\lambda \notin \sigma(T)$ if and only if $\|(T(t) - \lambda I(t))^{-1}\|$ is essentially bounded. Thus the set $K = \bigcap\{\overline{\bigcup_{t \in \delta} \sigma(T(t))}: \delta$ has full measure$\}$ is a compact subset of $\sigma(T)$. Since K is the intersection of compact sets, there exists a δ of full measure so that $K = \overline{\bigcup_{t \in \delta} \sigma(T(t))}$. Using the above theorem and proposition we can show that K is the spectrum of a quasi-similarity of T.

Proposition. *Let T be a decomposable operator and K as above. There exists a decomposable operator S which is quasi-similar to T and such that $\sigma(S) = K$.*

Proof. We shall show that T is the direct sum of operators each with spectrum in K. Let

$$g_{nm}(t) = \sup\left\{\left(\frac{n}{\operatorname{dist}(\lambda, K)} - \left\|(T(t) - \lambda I)^{-1}\right\|\right): \operatorname{dist}(\lambda, K) \geqslant \frac{1}{m}\right\}$$

and $f_{nm}(t) = \min(0, g_{nm}(t))$. From our theorem and corollary it follows that $\lim g_{nm}(t) = 0$ as $n \to \infty$ for each m and t. Choose $\varepsilon > 0$. By Egoroff's theorem there exists a set $A_{m\varepsilon}$ so that $f_{nm}(t) \to 0$ uniformly as $n \to \infty$ for t not in $A_{m\varepsilon}$ and $\mu(A_{m\varepsilon}) < \varepsilon/2^m$. Let $A_\varepsilon = \bigcup A_{m\varepsilon}$; then off A_ε, $f_{nm}(t) \to 0$ uniformly as $n \to \infty$ for all m. Let $B_\varepsilon = \Lambda \setminus A_\varepsilon$ and $T_\varepsilon = \int_{B_\varepsilon}^{\oplus} T(t)\,\mu(dt)$. It follows that $\sigma(T_\varepsilon) \subset K$ since $\|(T(t) - \lambda I)^{-1}\|$ is uniformly bounded for t in B_ε and λ not in K.

By choosing $\varepsilon = 1/k$, and using an induction argument, it follows that $T = \Sigma \oplus T_k$ with $\sigma(T_k) \subset K$. Moreover, we have $T_k = \int_{B_k}^{\oplus} T(t)\,\mu\, dt$, where the $\{B_k\}$ are disjoint measurable subsets of Λ. It follows from the previous proposition that $T = \Sigma \oplus T_k$ is quasi-similar to $S = \Sigma \oplus S_k$, and $\sigma(S) = \overline{\bigcup \sigma(S_k)} = \overline{\bigcup \sigma(T_k)}$ because S_k is similar to T_k. From the proof of theorem it follows that S_k and the implementing similarities R_k belong to the von Neumann algebra generated by T_k. Thus $S_k = \int_{B_k}^{\oplus} S_k(t)\,\mu(dt)$ with $S_k(t)$ similar to $T_k(t) = T(t)$ for t in B_k and, moreover, the operator R_k is also decomposable. Consequently, $S = \Sigma \oplus S_k$ as well as the operators $\Sigma \oplus R_k/\|R_k\|$ and $\Sigma \oplus R_k^{-1}/\|R_k^{-1}\|$ which implement the quasi-similarity of S on T are all decomposable with respect to the given decomposition of H.

Remark. Let $\{M_n\}$ be a sequence of invariant subspaces of T. C. Apostol calls M_n *basic* for T if for all n the subspaces M_n and $\overline{\bigvee_{m \neq n} M_m}$ are complementary and $\bigcap_n \overline{\bigvee_{m \geqslant n} M_m} = \{0\}$. If $T_n = T/M_n$, then it is easy to show that T is quasi-similar to $\Sigma \oplus T_n$ [1]. If $\overline{\bigcup \sigma(T_n)} = K$, then, as a corollary to the above results, we obtain that T is quasi-similar to an operator S with $\sigma(S) \subset K$. In particular, if T/M_n is quasi-nilpotent for all n, then T is quasi-similar to a quasi-nilpotent operator.

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Department of Mathematics, University of Nebraska, Lincoln, Nebraska 68588

Current address: Department of Mathematics, University of Nebraska, Lincoln, Nebraska 68588