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ON A CLASSIFICATION OF PLANE DOMAINS FOR HARDY CLASSES

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ABSTRACT. For every positive nubmer p, let O_p denote the class of plane domains W for which the Hardy class $H_p(W)$ contains no nonconstant functions, and $O_p^- = \bigcup\{O_q: 0 < q < p\}$. In this paper it is proved that O_p strictly contains O_p^- if $p > 1$.

1. Introduction. Let W be a domain in the extended complex plane S. For every positive number p, let $H_p(W)$ be the class of all single-valued analytic functions f in W for which $|f|^p$ admits a harmonic majorant there. Let O_p denote the class of W such that $H_p(W)$ contains no functions but the constants. Set $O_p^- = \bigcup\{O_q: 0 < q < p\}$. In this paper we deal with the inclusion relations among these classes of plane domains. The corresponding problem for arbitrary Riemann surfaces is completely solved by Heins **[2]**. For plane domains, however, the best results that we know by now are the following, one of which is due to Hejhal **[4]**, and the other to the author **[5]**:

THEOREM A (HEJHAL). *For* $n = 2, 3, \ldots,$

(1) $$O_{n/2}^- < O_{n/2}.$$

THEOREM B (KOBAYASHI). *For* $n = 2, 3, \ldots,$ *and* $p > n/2$,

(2) $$O_{n/2} < O_p.$$

Here $<$ means a strict inclusion relation.

The object of this paper is to prove the following theorem, which evidently includes the above two.

MAIN THEOREM. *If* $p \geqslant 1$, *then*

(3) $$O_p^- < O_p.$$

COROLLARY. *If* $p \geqslant 1$ *and* $q < p$, *then*

(4) $$O_q < O_p.$$

The Corollary is an easy consequence of the Main Theorem, since $O_q \leqslant O_p^-$ if $q < p$.

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2. Hejhal's lemmas. In this section we state two lemmas, which were proved by Hejhal [**4**, Theorems 10 and 17]. As for Lemma 2, however, we give another proof which is more elementary than that of Hejhal's and essentially due to Shimbo [**6**].

LEMMA 1. *Let* $c_1, \dots, c_m$ *be positive numbers with* $c_1 + \cdots + c_m = 2$, $m \geqslant 2$. *Suppose that E is a compact totally disconnected set of linear measure* 0 *which lies on an m-star formed by m rays emanating from the origin to the point at infinity, with successive angles* $\pi c_1, \dots, \pi c_m$. *Let* $c_0 = \max\{c_j\colon j = 1, \dots, m\}$ *and* $p \geqslant 1$. *Suppose that* $f \in H_p(S - E)$; *then*

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^{-\nu} \qquad (0 < |z| < \infty),$$

with $a_\nu = 0$ *for all* $\nu \geqslant 1/pc_0$.

For the proof see Hejhal [**4**, pp. 15–18].

LEMMA 2. *There exists a compact totally disconnected set F such that* (i) *F lies on the real axis*; (ii) *F is symmetric with respect to the origin*; (iii) *F is of linear measure* 0; *and* (iv) $z^{-1} \in H_p(S - F)$ *for all p with* $0 < p < 1$.

PROOF. Let A be a compact totally disconnected set such that (a) A lies on the interval $[-1, 1]$; (b) A is symmetric with respect to the origin; (c) $0 \notin A$; (d) A is of linear measure 0; and (e) A is of logarithmic capacity positive. For example, we can take the Cantor ternary set. Let $B = \bigcup_{k=-\infty}^{\infty} A + 2k$ and F_1 be the image of B under the map $h(z) = z^{-1}$. Finally let $F = F_1 \cup \{0\}$. We shall prove that F satisfies (i), ..., (iv) of the lemma. All except (iv) are trivial. Since h maps $S - F$ conformally onto $\mathbf{C} - B$, we must show that $z \in H_p(\mathbf{C} - B)$ for all p with $0 < p < 1$. It is well known that every analytic function f for which $|\operatorname{Im} f|$ admits a harmonic majorant is of class H_p for all p with $0 < p < 1$ (see [**1**, p. 35]). Therefore it is sufficient to prove that $|\operatorname{Im} z|$ admits a harmonic majorant in $\mathbf{C} - B$. Let ω be the harmonic function in $(\mathbf{C} - B) \cap \{z\colon \operatorname{Im} z < 2\}$ with boundary value 1 on the line $\{z\colon \operatorname{Im} z = 2\}$ and 0 on B. Since $\operatorname{Cap}(B) > 0$, we see that ω is nonconstant. Since B is invariant under the translation $\phi(z) = z + 2$, so is ω. Hence,

$$\begin{aligned}\sup\{\omega(z)\colon \operatorname{Im} z = 1\} &= \sup\{\omega(z)\colon \operatorname{Im} z = 1 \text{ and } -1 \leqslant \operatorname{Re} z \leqslant 1\}\\ &< 1 - \varepsilon\end{aligned}$$

for some ε with $0 < \varepsilon < \frac{1}{2}$. Therefore

$$\varepsilon^{-1}\omega(z) < \operatorname{Im} z + \varepsilon^{-1} - 2 \tag{5}$$

on the line $\{z\colon \operatorname{Im} z = 1\}$ and, hence, in $\{z\colon 1 < \operatorname{Im} z < 2\}$ by the maximum principle. We define s in $\mathbf{C} - B$ as follows:

$$s(z) = \begin{cases} \operatorname{Im} z + \varepsilon^{-1} & \text{if } \operatorname{Im} z \geqslant 2,\\ \varepsilon^{-1}\omega(z) + 2 & \text{if } \operatorname{Im} z < 2.\end{cases} \tag{6}$$

We shall show that s is superharmonic in $\mathbf{C} - B$. To show this, it is sufficient to prove that s is superharmonic on the line $\{z\colon \operatorname{Im} z = 2\}$. But this follows easily from (5) (see the proof of Lemma 3 below). Then we see that $s(z) + s(\bar{z})$ serves as a superharmonic majorant of $|\operatorname{Im} z|$ in $\mathbf{C} - B$. Since $|\operatorname{Im} z|$ is subharmonic in $\mathbf{C} - B$, it admits a harmonic majorant there, as desired.

3. Proof of the Main Theorem. We must construct a plane domain W such that $W \in O_p - O_p^-$. Let F be as in Lemma 2, and ψ be the conformal map of $\{\zeta\colon |\arg \zeta| < \pi\}$ onto $\{z\colon |\arg z| < \pi/p\}$ defined by the branch of $z = \zeta^{1/p}$ such that $\psi(1) = 1$. Let E_0 be the image of $F \cap \{\zeta\colon \operatorname{Re} \zeta > 0\}$ under ψ. Let $c_1, \ldots, c_m$ be positive numbers with $c_1 + \cdots + c_m = 2$, $m \geqslant 2$, such that $c_0 = \max\{c_j\colon j = 1, \ldots, m\} = 1/p$. Let $E_1 = \bigcup_{\nu=1}^m e^{i\theta_\nu} E_0$, where $\theta_\nu = \pi \sum_{j=1}^{\nu} c_j$. Finally, let $E = E_1 \cup \{0\}$ and $W = S - E$. On applying Lemma 2, it turns out that $H_p(W)$ contains no nonconstant functions, i.e., $W \in O_p$.

Next we must show that $H_q(W)$ contains a nonconstant function for any $q < p$. Let q be fixed with $0 < q < p$. We shall prove that $f(z) = z^{-1}$ belongs to $H_q(W)$. Let $w(z) = |z|^{-q}$. We must show that w admits a harmonic majorant in W. Let λ be the least harmonic majorant of $w_1(\zeta) = |\zeta|^{-q/p}$ in $S - F$. We define χ in W as follows:

$$\chi(z) = \lambda\big(\psi^{-1}(e^{-i\theta_\nu} z)\big) + g(z) \quad \text{if } \theta_\nu - \frac{\pi}{2}\, c_\nu \leqslant \arg z < \theta_\nu + \frac{\pi}{2}\, c_{\nu+1}$$

for $\nu = 1, \ldots, m$, where $c_{m+1} = c_1$, and g is the Green's function of W with pole at ∞. We shall show that χ serves as a superharmonic majorant of w in W. It is trivial that $\chi \geqslant w$ in W and that χ is superharmonic in W except on the m-star $\{z\colon \arg z = \theta_\nu - (\pi/2)c_\nu, \nu = 1, \ldots, m\}$. The following lemma shows the superharmonicity of χ on the m-star.

LEMMA 3. *Let* $0 < \theta < \pi$. *Define*

$$\mu(z) = \begin{cases} \lambda(z) & \text{if } 0 < \arg z \leqslant \theta/2, \\ \lambda(e^{-i\theta} z) & \text{if } \theta/2 < \arg z < \theta; \end{cases}$$

then μ *is superharmonic on the line* $\{z\colon \arg z = \theta/2\}$.

PROOF. First we shall prove that

$$\mu \leqslant \lambda \quad \text{in } \{z\colon 0 < \arg z < \theta\}. \tag{7}$$

Let $\{I_k\}$ be a decreasing sequence of compact sets on the real axis such that (a) I_k is a union of a finite number of closed intervals; (b) I_k is symmetric with respect to the origin; (c) $I_k \supset F$ and (d) $F = \bigcap I_k$. Let λ_k be the least harmonic majorant of w_1 in $S - I_k$. It is well known that λ_k converges to λ uniformly on every compact subset of $S - F$. Let

$$\phi_k(z) = \lambda_k(z) - \lambda_k(e^{-i\theta} z),$$

then ϕ_k is harmonic in $S - (I_k \cup e^{i\theta} I_k)$. For every $x \in I_k$, we easily see

$$\phi_k(x) = \lambda_k(x) - \lambda_k(xe^{-i\theta}) = w_1(x) - \lambda_k(xe^{i\theta}) \leqslant 0$$

and

$$\phi_k(xe^{i\theta}) = \lambda_k(xe^{i\theta}) - \lambda_k(x) = \lambda_k(xe^{i\theta}) - w_1(x) = -\phi_k(x),$$

since λ_k is symmetric with respect to the real axis. By symmetry, $\phi_k = 0$ on the 4-star $\{z\colon \arg z = \theta/2 + j\pi/2, j = 0, 1, 2, 3\}$. Applying the maximum principle, we see that $\phi_k \geqslant 0$ in $\{z\colon \theta/2 < \arg z < \theta/2 + \pi/2\}$. By letting $k \to \infty$, we get (7).

Let z_0 be any point with $\arg z_0 = \theta/2$, and let r be a small positive number. Using (7) we see that

$$\mu(z_0) = \lambda(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(z_0 + re^{i\alpha})\,d\alpha$$

$$\geqslant \frac{1}{2\pi}\int_0^{2\pi} \mu(z_0 + re^{i\alpha})\,d\alpha,$$

and, hence, μ is superharmonic at z_0 as desired.

ADDED IN PROOF. Recently M. Hasumi [7] has completely solved the H_p classification problem for plane domains, which we treated in this paper, in more general form than Heins [2] solved the problem for arbitrary Riemann surfaces.

REFERENCES

1. P. L. Duren, *Theory of H_p spaces*, Academic Press, New York, 1970.
2. M. Heins, *Hardy classes on Riemann surfaces*, Lecture Notes in Math., vol. 98, Springer-Verlag, Berlin and New York, 1969.
3. D. A. Hejhal, *Classification theory for Hardy classes of analytic functions*, Bull. Amer. Math. Soc. 77 (1971), 767–771.
4. ______, *Classification theory for Hardy classes of analytic functions*, Ann. Acad. Sci. Fenn. Ser. A I No. 566 (1973), 1–28.
5. S. Kobayashi, *On H_p classification of plane domains*, Kōdai Math. Sem. Rep. **27** (1976), 458–463.
6. T. Shimbo, *On harmonic majoration*, Kōdai Math. Sem. Rep. **28** (1977), 278–283.
7. M. Hasumi, *Hardy classes on plane domains*, preprint, Inst. Mittag-Leffler Rep. **2** (1977) (to appear).

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