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PRODUCTS OF SINGULAR CONTINUOUS MEASURES

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ABSTRACT. Let G be a compact Abelian group. It is shown that if the Fourier transform of $f \in A(G)$ satisfies certain lacunary conditions, then f may be factored as the convolution product of singular continuous measures.

Let G be a compact Abelian group with dual group Γ. Doss [3] has shown that if μ is any continuous measure on G, then there exists a singular measure $\nu \neq 0$ on G such that $(\mu * \nu)\hat{} \in L^1(\Gamma)$, i.e., the Radon-Nikodým derivative of $\mu * \nu$ has an absolutely convergent Fourier series. Extensions of this theorem to locally compact groups are given in [4]. In this paper we prove a partial converse to Doss's theorem. We show (Theorem 1) that if f has an absolutely convergent Fourier series and $\hat{f}$ satisfies certain lacunary conditions, then there are singular continuous measures μ and ν such that $f = \mu * \nu$ and $\|\mu\| \|\nu\| < \|\hat{f}\|_1$. We then list some consequences of this theorem and give some examples.

Throughout G will denote a compact Abelian group with dual Γ; $M(G)$ the complex regular Borel measures on G; $A(G) = \{f \in C(G): \hat{f} \in L^1(\Gamma)\}$ with $\|f\|_A = \|\hat{f}\|_1$. For $\mu \in M(G)$ we denote the support of $\hat{\mu}$ by $S(\hat{\mu}) = \{\gamma \in \Gamma: \hat{\mu}(\gamma) \neq 0\}$. We follow the exposition on dissociate sets and Riesz products given in [2]. In particular, $\Theta \subset \Gamma$ is dissociate if $0 \notin \Theta$ and each $\gamma \in \Gamma$ can be written in at most one way (except for order of summands) as

$$\gamma = \sum_{1}^{n} \varepsilon_j \gamma_j \tag{1}$$

where the γ_j $(1 \leqslant j \leqslant n)$ are distinct elements of Θ, $\varepsilon_j \in \{\pm 1\}$ if $2\gamma_j \neq 0$, and $\varepsilon_j = 1$ if $2\gamma_j = 0$. $\Omega(\Theta)$ consists of 0 and all characters of the form (1). We will make use of the following: if μ is a Riesz product generated by b, Θ where, say, $|b(\theta)| \leqslant \frac{1}{2}$, $b(\theta) \neq 0$ for all $\theta \in \Theta$, then μ is singular if $\Sigma_\Theta |b(\theta)|^2 = \infty$ and μ is continuous if $\Sigma_\Theta(1 - |b(\theta)|) = \infty$. These facts and much additional information may be found in [1], [2], [6], [8], and [10].

THEOREM 1. *Let $f \in A(G)$ and suppose there exists an infinite dissociate set $\Theta \subset \Gamma$ such that $[S(\hat{f}) - S(\hat{f})] \cap \Omega(\Theta)$ is finite. Then there exist singular continuous measures μ and ν on G such that $f = \mu * \nu$ and $\|\mu\| \|\nu\| \leqslant \|f\|_A$.*

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PROOF. By removing a finite set from Θ we may assume without loss of generality that

$$\left[S(\hat{f}) - S(\hat{f})\right] \cap \Omega(\Theta) = \{0\}.$$

Write $\Theta = \Psi \cup \Lambda$ where both Ψ and Λ are infinite and $\Psi \cap \Lambda = \varnothing$. Choose an increasing sequence (F_n) of finite subsets of Γ such that $S(\hat{f}) = \bigcup_1^\infty F_n$. Next write

$$\hat{f}(\gamma) = |\hat{f}(\gamma)|^{1/2} |\hat{f}(\gamma)|^{1/2} \operatorname{sgn} \bar{\hat{f}}(\gamma).$$

Then

$$f = g * h \quad \text{and} \quad \|f\|_A = \|g\|_2 \, \|h\|_2$$

where $g, h \in L^2(G)$ are defined by $\hat{g} = |\hat{f}|^{1/2}$ and $\hat{h} = |\hat{f}|^{1/2} \operatorname{sgn}(\bar{\hat{f}})$.

Let σ be any singular continuous Riesz product generated by b, Ψ (e.g., $b(\theta) \equiv \frac{1}{2}$). σ is a probability measure with $S(\hat{\sigma}) \subset \Omega(\Psi) \subset \Omega(\Theta)$. In particular, (1) implies that

$$\hat{\sigma}(\gamma - \chi) = 0 \quad \text{for all } \gamma, \chi \in S(\hat{f}), \gamma \neq \chi. \tag{2}$$

Thus, setting $Q_n = \sum_{F_n} \hat{g}(\gamma)\gamma$ $(n \geq 1)$ we have for $m \geq n$

$$\int_G |Q_m - Q_n|^2 d\sigma = \sum_{F_m/F_n}\sum \hat{g}(\gamma)\overline{\hat{g}(\chi)}\hat{\sigma}(\chi - \gamma) = \sum_{F_m/F_n} |\hat{g}(\gamma)|^2.$$

It follows that the sequence (Q_n) is Cauchy in $L^2(\sigma)$. Let g' denote the L^2 limit of the Q_n and let $\mu \in M(G)$ be the measure whose Radon-Nikodým derivative with respect to σ is g': $\mu = g'\sigma$. Then μ is singular and continuous since σ is. Further, (Q_n) converges weakly to g' in $L^2(\sigma)$ so that

$$\hat{\mu}(\chi) = \lim_{n\to\infty} \sum_{\gamma \in F_n} \hat{g}(\gamma)\hat{\sigma}(\chi - \gamma) \qquad (\chi \in \Gamma). \tag{3}$$

In particular, for $\chi \in S(\hat{f})$, (2) shows that $\hat{\mu}(\chi) = \hat{g}(\chi)$. Moreover,

$$\|\mu\| \leq \|g'\|_{2,\sigma} \leq \|\hat{g}\|_2 = \|g\|_2.$$

Next, let τ be any singular continuous Riesz product on Λ, set $P_n = \sum_{F_n} \hat{h}(\gamma)\gamma$ and repeat the above argument to obtain a singular continuous measure $\nu = h'\tau$ such that $\hat{\nu} = \hat{h}$ on $S(\hat{f})$ and $\|\mu\| \leq \|h\|_2$.

Suppose $(\mu * \nu)\hat{}(\chi) \neq 0$. It follows from (3) that $\hat{\sigma}(\chi - \gamma) \neq 0$ for some $\gamma \in S(\hat{f})$, whence, $\chi = \gamma + \omega$ for some $\gamma \in S(\hat{f})$, $\omega \in \Omega(\Psi)$. Similarly, $\chi = \gamma' + \omega'$ for some $\gamma' \in S(\hat{f})$ and $\omega' \in \Omega(\Lambda)$. Now $\Psi \cap \Lambda = \varnothing$ and Θ is dissociate. This shows first that $\gamma - \gamma' = \omega' - \omega \in \Omega(\Theta)$ and then in combination with (1) implies that $\omega = \omega' = 0$, i.e., $\chi = \gamma = \gamma' \in S(\hat{f})$. Thus, $(\mu * \nu)\hat{}$ vanishes off $S(\hat{f})$. On the other hand, from above $(\mu * \nu)\hat{} = (g * h)\hat{} = \hat{f}$ on $S(\hat{f})$. Thus, $f = \mu * \nu$. Finally, $\|\mu\| \, \|\nu\| \leq \|g\|_2 \, \|h\|_2 = \|f\|_A$.

For $\mathfrak{F} \subset A(G)$ and $E \subset \Gamma$, put $\mathfrak{F}_E = \{f \in \mathfrak{F}\colon S(\hat{f}) \subset E\}$. We say that $f \in \mathfrak{F}_E$ admits singular continuous factorization if there exist singular continuous measures μ and ν such that $f = \mu * \nu$ and $\|\mu\| \, \|\nu\| \leq \|f\|_A$.

COROLLARY 1. *If $E \subset \Gamma$ is a Sidon set such that $(E - E) \cap \Omega(\Theta)$ is finite for some infinite dissociate set Θ, then each $f \in L_E^\infty(G)$ admits singular continuous factorization.*

COROLLARY 2. *Let $E \subset \Gamma$. If there exists a continuous measure $\sigma \in M(G)$ such that $|\hat{\sigma}| \geqslant \delta > 0$ on $E - E$, then each $f \in A_E(G)$ admits singular continuous factorization.*

PROOF. Corollary 1 is clear. Corollary 2 follows from Lemma 1, Doss **[3]** (or Lemma 5, **[4]**): if σ is continuous and $\delta > 0$, then there exists an infinite dissociate set Θ such that $|\hat{\sigma}(\omega)| < \delta$ for all $\omega \in \Omega(\Theta) \setminus \{0\}$. Thus, $(E - E) \cap \Omega(\Theta) = \{0\}$.

REMARKS. Every dissociate set $\Theta \subset \Gamma$ satisfies the hypothesis of Corollary 2: If μ is the continuous Riesz product with $\hat{\mu}(\theta) = \frac{1}{2}$ for every $\theta \in \Theta$, then $\hat{\mu}(\theta_1 - \theta_2) \geqslant \frac{1}{4}$ for all $\theta_1, \theta_2 \in \Theta$.

Every Sidon set E satisfies $E \cap \Omega(\Theta) = \{0\}$ for some infinite dissociate set Θ. For by the Hartman-Wells Theorem [7, 4.8] there is a continuous measure μ such that $\hat{\mu} = 1$ on E, hence, the lemma of Doss cited above provides the required Θ. Of course, this argument fails for $E - E$ since this set is not Sidon. Indeed, not every Sidon set satisfies the hypothesis of Corollary 1; $E = \{4^n, 4^n + n\colon n \geqslant 1\}$ is a Sidon set for which $E - E = Z$. On the other hand, there are rather "large" sets E satisfying $(E - E) \cap \Omega(\Theta) = \{0\}$. For example, one can construct infinite dissociate sets Θ_1 and Θ_2 for which $[\Omega(\Theta_1) - \Omega(\Theta_1)] \cap \Omega(\Theta_2) = \{0\}$. Since $\Omega(\Theta_1)$ supports transforms of nonzero singular measures this shows, in particular, that there are non-Riesz sets E (E is a Riesz set if $S(\hat{\mu}) \subset E$ implies $\mu \in M_a(G)$) for which every $f \in A_E(G)$ admits singular continuous factorization. It may be the case, in fact, that every $f \in A(G)$ (perhaps even $f \in L^1(G)$) admits such a factorization. Little appears to be known in this direction. It is known that not every continuous measure can be factored as a product of continuous measures **[9]**.

COROLLARY 3. *Let H be an infinite closed subgroup of G. Then each $f \in A(G)$ that is constant on cosets of H admits singular continuous factorization. In particular, each continuous positive definite function on G for which $\{x \in G\colon f(x) = f(0)\}$ is infinite admits such a factorization.*

PROOF. H is not discrete so Haar measure m on H may be regarded as a continuous measure in $M(G)$. Since $\hat{m}$ is the characteristic function of the annihilator $E = \{\gamma \in \Gamma\colon \gamma = 1 \text{ on } H\}$ of H, Corollary 2 shows that each $f \in A(G)$ with $S(\hat{f}) \subset E - E = E$ admits the asserted factorization. These functions are exactly those of the form $g \circ \pi$ where $g \in A(G/H)$ and $\pi\colon G \to G/H$ is the quotient map, i.e., they are the members of $A(G)$ that are constant on cosets of H. Finally, if f is continuous positive definite on G, then $H = \{x \in G\colon f(x) = f(0)\}$ is a closed subgroup upon whose cosets f is constant.

For $G = T$ it is possible to construct a large collection of sets $E \subset Z$ for which each $f \in A_E$ admits singular continuous factorization. We begin with a definition and a lemma whose proof is straightforward and omitted.

A set $E \subset Z$ is said to have (arbitrarily) large symmetric gaps relative to I_n if there exist intervals $I_n = [a_n, b_n]$ where a_n, b_n are positive integers with $a_n < b_n < a_{n+1}$ such that $\lim_{n\to\infty}(b_n - a_n) = \infty$ and

$$(*)\qquad \liminf_{n\to\infty} n^{-1}\left|E \cap \bigcup_1^n (-I_k \cup I_k)\right| < 1$$

($|A|$ = cardinality of A). We note that this occurs, for instance, if there exists a function $g > 0$ such that

$$\liminf_{n\to\infty} n^{-1}g(n) = 0$$

and $|E \cap \bigcup_1^n(-I_k \cup I_k)| \leqslant \rho g(n)$ $(n \geqslant 1)$ for some constant $\rho > 0$. In particular, Sidon sets $(g(n) = \log n)$ and, more generally, Λ_p sets $(p > 2)$ $(g(n) = n^{2/p})$ have large symmetric gaps [5, 37.25].

Lemma. *Let $(x_k)_1^\infty$ be a sequence of positive integers with $x_k \geqslant 3$ for all $k \geqslant 2$. Let $\theta_n = \prod_1^n x_k$, $\Theta = (\theta_n)_1^\infty$, and set*

$$\Omega_n = \left[\theta_n, \sum_1^n \theta_k\right] \cup \left[\theta_{n+1} - \sum_1^n \theta_k, \theta_{n+1} - 1\right] \qquad (n \geqslant 1).$$

Then $\theta_{n+1} > 2\sum_1^n \theta_k$ $(n \geqslant 1)$ so that Θ is dissociate and

$$\Omega(\Theta) \subset \left(\bigcup_1^\infty \Omega_n\right) \cup \{0\} \cup \left(-\bigcup_1^\infty \Omega_n\right).$$

Theorem 2. *Let $E \subset Z$. If $E - E$ has large symmetric gaps relative to I_n, then each $f \in A_E(T)$ admits singular continuous factorization.*

Proof. Let $I_n = [a_n, b_n]$ be as in the definition and for a sequence (x_n) of integers set $\theta_n = \prod_1^n x_k$, $\Theta = (\theta_n)_1^\infty$. For $n \geqslant 1$, let

$$F_n = \left[\theta_n - \sum_1^{n-1} \theta_k, \theta_n + \sum_1^{n-1} \theta_k\right].$$

It suffices by Theorem 1 to show that there is an infinite dissociate set Θ such that $(E - E) \cap \Omega(\Theta) = \{0\}$. In view of the Lemma we need only show that a sequence (x_n) can be chosen satisfying $x_n \geqslant 3$ $(n \geqslant 1)$ and $(E - E) \cap \bigcup_1^\infty (-F_n \cup F_n) = \varnothing$. We do this by induction. Let $x_1 \geqslant 3$ be any integer such that $\pm x_1 \notin E - E$. Then $(E - E) \cap \{\pm x_1\} = (E - E) \cap (-F_1 \cup F_1) = \varnothing$. Assume $x_k \geqslant 3$ $(1 \leqslant k \leqslant n)$ have been chosen so that $(E - E) \cap \bigcup_1^n (-F_k \cup F_k) = \varnothing$. The induction will be complete provided $x_{n+1} \geqslant 3$ can be selected such that $(E - E) \cap (-F_{n+1} \cup F_{n+1}) = \varnothing$. Suppose to the contrary that for each $m \geqslant 3$,

$$(1)\qquad (E - E) \cap (-H_m \cup H_m) \neq \varnothing$$

where

$$H_m = \left[\theta_n m - \sum_1^n \theta_k, \theta_n m + \sum_1^n \theta_k\right].$$

Since $a_n < a_{n+1} (n \geq 1)$ and $\lim_{n\to\infty}(b_n - a_n) = \infty$ there is a N such that $\theta_n^{-1}(a_j + \Sigma_1^n \theta_k) \geq 3$ and $b_j - a_j > \theta_n + 2\Sigma_1^n \theta_k$ for all $j \geq N$. Then

$$\theta_n^{-1}\left(b_j - \sum_1^n \theta_k\right) - \theta_n^{-1}\left(a_j + \sum_1^n \theta_k\right) > 1.$$

Hence, for each $j \geq N$ there is an integer $m_j \geq 3$ satisfying

(2) $$\theta_n^{-1}\left(a_j + \sum_1^n \theta_k\right) \leq m_j \leq \theta_n^{-1}\left(b_j - \sum_1^n \theta_k\right).$$

It follows that

(3) $$H_{m_j} = \left[\theta_n m_j - \sum_1^n \theta_k, \theta_n m_j + \sum_1^n \theta_k\right] \subset [a_j, b_j].$$

Further, from (2)

$$\theta_n m_j + \sum_1^n \theta_k \leq b_j < a_{j+1} \leq \theta_n m_{j+1} - \sum_1^n \theta_k$$

so that the intervals H_{m_j} in (3) are pairwise disjoint. Thus, the assumption in (1) leads to

$$M - N + 1 \leq \left|(E - E) \cap \bigcup_{j=N}^{M} (-H_{m_j} \cup H_{m_j})\right|$$

$$\leq \left|(E - E) \cap \bigcup_{j=1}^{M} (-I_j \cup I_j)\right|$$

for all $M \geq N$. This contradicts line (*) above and completes the proof.

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