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SIMPLICIAL SCHREIER SYSTEMS AND THE COMMUTATOR SUBGROUP OF THE FREE GROUP ON THE CIRCLE

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ABSTRACT. It is shown that the commutator subgroup of the free simplicial group on the circle has a simplicial Schreier system and is a free simplicial group on a pointed simplicial set.

Introduction. Subgroups of a free topological (simplicial) group need not be free topological (simplicial) **[6]**, **[9]**, **[2]**, **[3]**. In **[3]** it is proved that the commutator subgroup $[FS^n, FS^n]$ of FS^n $(n > 1)$ is not free topological. The homotopy theoretic methods used there were not helpful for S^1. In this paper, we use simplicial methods to construct a simplicial Schreier system for the subgroup $[FS^1, FS^1]$ in FS^1 and show that the free simplicial basis for $[FS^1, FS^1]$ is of the same homotopy type as S^2.

The referee has pointed out that the term "free simplicial group" has been used in two conflicting senses in the literature. Firstly, in the sense of Kan's and Whitehead's paper **[10]** where these groups are free in each dimension with generators which are stable under degeneracies but not necessarily under face maps. Secondly, in the sense of [2], and also **[12]**, where these groups are free in each dimension with generators which are stable under both degeneracies and face maps. However, in our results the term "free simplicial group on a pointed simplicial set" is used in the second sense.

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1. **Simplicial Schreier systems.** Let K be a pointed simplicial set, $F = FK$ be the free simplicial group on K as defined by J. Milnor **[11]** (see also the appendix of [2]). The face and degeneracy operators d_i, s_i are the homomorphic extensions of those in K. Let $K \to^{\eta} F$ be the inclusion and $H \subset F$ be a simplicial subgroup.

DEFINITION 1.1. A (two-sided) simplicial Schreier system for the cosets of a (normal) subgroup H in F is a subcomplex S of F such that for each $n \geqslant 0$, S_n is a (two-sided) Schreier system (Hall [7]) for the cosets of H_n in F_n. That is:

(i) each coset of F_n/H_n contains exactly one element of S_n;

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(ii) the identity $e_n \in S_n$;

(iii) if $x_{i_1}^{\varepsilon_1} \cdots x_{i_m}^{\varepsilon_m}$ is a reduced word in S_n, then $x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ for $1 \leqslant k < m$ (and $x_{i_h}^{\varepsilon_h} \cdots x_{i_k}^{\varepsilon_k}$ for $1 \leqslant h \leqslant k \leqslant m$) is again in S_n.

PROPOSITION 1.2. *If S is a simplicial Schreier system for the cosets of H in F, then the map φ: $F/H \to F$, which sends a coset to its unique representative in S, is a simplicial section of the natural map π: $F \to F/H$.*

PROOF. For $x \in F_n$, let $\varphi_n(H_n x) = a \in S_n$. Then $H_n x \cap S_n = \{a\}$. Let $H_{n-1}(d_i x) \cap S_{n-1} = \{b\} \subset S_{n-1}$. Then $d_i(H_n x \cap S_n) \subset H_{n-1}(d_i x) \cap S_{n-1} = \{b\}$, i.e. $d_i a = b$. Hence $d_i \varphi_n(H_n x) = \varphi_{n-1} d_i(H_n x)$. Similarly $s_i \varphi_n = \varphi_{n+1} s_i$. Clearly $\pi_n \varphi_n = \text{id}$ on F_n/H_n. □

THEOREM 1.3 (SIMPLICIAL NIELSEN-SCHREIER THEOREM). *If $S = \{\varphi(Hx)$: $x \in F\}$ is a simplicial Schreier system for the cosets of H in F, then there exists a subcomplex L of F such that $H = FL$.*

PROOF. Let $L_n = \{yx\varphi_n(Hyx)^{-1}$: $y \in S_n, x \in K_n\}$. The usual Nielsen-Schreier Theorem (Hall [7]) shows that $L_n \setminus \{e_n\}$ is an algebraically free basis for H_n for each $n \geqslant 0$.

Since φ is simplicial, by Proposition 1.2, it follows that L is stable under face and degeneracy operators, i.e. L is a subcomplex of F for which $H = FL$. □

REMARK 1.4. Simplicial Schreier systems do not always exist, contrary to the algebraic case (Hall [7]) since simplicial subgroups of free simplicial groups are not always free simplicial ([6], [9], and [2]). For example, F. Clarke [3] has shown that the commutator subgroup $[FS^n, FS^n]$ of FS^n is not free for $n > 1$. In fact it is easy to see in this case that there is no simplicial section of the natural projection

$$FS^n \xrightarrow{\pi} FS^n/[FS^n, FS^n] \cong AS^n$$

where AS^n is the free simplicial abelian group on the n-sphere, S^n. For this would imply that

$$H^*(AS^n; \mathbf{Z}/2) \xrightarrow{\pi^*} H^*(FS^n; \mathbf{Z}/2)$$

was a monomorphism. But AS^n is a $K(\mathbf{Z}, n)$ ([11, Theorem 24.5] or [1, Theorem 5.12]), FS^n has the homotopy type of ΩS^{n+1} [8], and π^* maps the fundamental class $x \in H^n(K(\mathbf{Z}, n); \mathbf{Z}/2)$ to the generator $y \in H^n(\Omega S^{n+1}; \mathbf{Z}/2)$. Now $x^2 \neq 0$ for $n > 1$ but $y^2 = 0$, so $\pi^*(x^2) = 0$.

Even if $H \subset F$ is a free simplicial subgroup, then H may not have a simplicial Schreier system; this can be shown as follows: Let $I = [0, 1]$ be the unit interval considered as a simplicial set with three nondegenerate simplices, a in dimension 1 and e, f in dimension 0 with $d_0 a = e$, $d_1 a = f$; e is the base point. Consider $H = F\{e, f\} \subset FI$, the free simplicial subgroup of FI generated by f. Suppose $S \subset FI$ is a simplicial Schreier system for H in FI; then, by Proposition 1.2, the map φ is a simplicial section of the natural map $FI \to^{\pi} FI/H$. This would imply that $FI \cong H \times FI/H$ (as simplicial sets) and, hence, that FI is not connected, which is a contradiction. Hence the existence of a simplicial Schreier system is not a necessary condition for freeness of a subgroup. This agrees with the case of a topological Schreier transversal [13].

On the other hand if G is a simplicial group and $FG \xrightarrow{p} G$ is the unique homomorphism such that $p\eta = \mathrm{id}$, then clearly $\eta G \subset FG$ is a simplicial Schreier system for the kernel of p. In fact $\ker p \cong F(G \wedge G)$ [5].

We can construct a simplicial Schreier system for the commutator subgroup of FK if K can be simplicially ordered in the following way:

DEFINITION 1.5. Let K be a pointed simplicial set with base point $e_n = s_0^n e$. Then K is said to be simplicially ordered if there exists a total ordering on $K_n \setminus \{e_n\}$ such that $K_n \setminus \{e_n\} \xrightarrow{s_i} K_{n+1} \setminus \{e_{n+1}\}$ is order preserving and if $x < y$ in $K_n \setminus \{e_n\}$ implies $d_i x \leqslant d_i y$ unless $d_i x = e_{n-1}$ or $d_i y = e_{n-1}$.

In §2 we will construct a simplicial ordering on the 1-sphere, S^1. Here we have the following results:

PROPOSITION 1.6. *If $\{K_\lambda;\ \lambda \in \Lambda\}$ is a family of pointed simplicial sets each with a simplicial ordering, then $\bigvee_{\lambda \in \Lambda} K_\lambda$ has a simplicial ordering.*

PROOF. $(\bigvee_{\lambda \in \Lambda} K_\lambda)_n \setminus \{e_n\} = \coprod_{\lambda \in \Lambda}((K_\lambda)_n \setminus \{e_n\})$. Totally order the indexing set Λ and define the ordering on $(\bigvee_{\lambda \in \Lambda} K_\lambda)_n \setminus \{e_n\}$ as follows: $x < y$ if

(i) $x, y \in (K_\lambda)_n \setminus \{e_n\}$ and $x < y$ in the given ordering on K_λ; or

(ii) $x \in (K_\lambda)_n \setminus \{e_n\}, y \in (K_\mu)_n \setminus \{e_n\}$ and $\lambda < \mu$ in the ordering on Λ. It is easy to check that this defines a simplicial ordering on $\bigvee_{\lambda \in \Lambda} K_\lambda$. □

THEOREM 1.7. *If K is a pointed simplicial set with a simplicial ordering, then there exists a simplicial Schreier system for the commutator subgroup $[FK, FK]$ of FK.*

PROOF. Since $(AK)_n \cong (FK)_n/[(FK)_n, (FK)_n]$ is the free abelian group on K_n, it follows that

$$S_n = \{x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} \in (FK)_n \colon x_1 < x_2 < \cdots < x_m \text{ in } K_n \setminus \{e_n\}, k_i \in \mathbf{Z}\}$$

is an algebraic, two-sided, Schreier system for $[(FK)_n, (FK)_n]$ in $(FK)_n$. Now applying the face operator d_i to $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$ we obtain $(d_i x_1)^{k_1} (d_i x_2)^{k_2} \cdots (d_i x_m)^{k_m}$ in $(FK)_{n-1}$. We can delete any $(d_i x_j)^{k_j}$ for which $d_i x_j = e_{n-1}$ to obtain an element of S_{n-1}, since $d_i x_j \leqslant d_i x_k$ if neither is the base point. Hence $d_i S_n \subset S_{n-1}$. Similarly $s_i S_n \subset S_{n+1}$, i.e. S is a subcomplex of FK and, hence, S is a simplicial Schreier system. □

Now Theorem 1.3 and 1.7 give

COROLLARY 1.8. *If K is a pointed simplicial set with a simplicial ordering, then the commutator subgroup of FK is a free simplicial group on a pointed simplicial set.*

2. **A free simplicial basis for** $[FS^1, FS^1]$. We will show how S^1 can be simplicially ordered, but before doing so we prove the following lemma:

LEMMA 2.1. *Let K be a simplicial set with nondegenerate a in K_1 and define $x_j = s_{n-1} \cdots \hat{s}_j \cdots s_0 a$ in K_n $(0 \leqslant j \leqslant n-1)$ where n denotes omission of the jth degeneracy. Then*

(i)

$$d_i x_j = \begin{cases} x_j & \text{if } i > j, j \neq n-1, \\ x_{j-1} & \text{if } i \leqslant j, j \neq 0, \end{cases}$$

while $d_0 x_0 = s_0^{n-1} d_0 a$ and $d_n x_{n-1} = s_0^{n-1} d_1 a$.

(ii)

$$s_i x_j = \begin{cases} x_j & \text{if } i > j, \\ x_{j+1} & \text{if } i \leqslant j. \end{cases}$$

PROOF. If $j \leqslant n-2$, then $x_j = s_{n-1} x_j$ in K_n. Hence

$$d_i x_j = d_i s_{n-1} x_j = \begin{cases} s_{n-2} d_i x_j & \text{if } i < n-1, \\ x_j & \text{if } i = n, n-1, \end{cases}$$

while if $x_{n-1} = s_{n-2} x_{n-2}$ in K_{n+1}, then

$$d_i x_{n-1} = d_i s_{n-2} x_{n-2} = \begin{cases} s_{n-3} d_i x_{n-2} & \text{if } i < n-2, \\ s_{n-2} d_{n-1} x_{n-2} & \text{if } i = n, \\ x_{n-2} & \text{if } i = n-2, n-2. \end{cases}$$

Clearly

$$d_0 x_0 = d_0 (s_{n-1} \cdots \hat{s}_0 a) = s_{n-2} \cdots s_0 d_0 a = s_0^{n-1} d_0 a$$

and

$$d_n x_{n-1} = d_n (\hat{s}_{n-1} \cdots s_0 a) = s_{n-2} \cdots s_0 d_1 a = s_0^{n-1} d_1 a.$$

Hence (i) is proved. A similar argument can be used to show (ii). □

THEOREM 2.2. *If S^1 denotes the pointed simplicial set with two nondegenerate simplices, e in dimension 0 and a in dimension 1, then S^1 has a simplicial ordering and, hence, $[FS^1, FS^1]$ is a free simplicial group on a pointed simplicial set.*

PROOF. The set of n-simplices in S^1 is

$$(S^1)_n = \{e, x_0, x_1, \ldots, x_{n-1}\}$$

where $e = s_{n-1} \cdots s_0 e$ and $x_j = s_{n-1} \cdots \hat{s}_j \cdots s_0 a$ $(0 \leqslant j \leqslant n-1)$ [**4**, p. 110]. Define an ordering on $(S^1)_n \setminus \{e\}$ as follows:

$$x_i < x_j \quad \text{if } i < j, \qquad i, j = 0, 1, \ldots, n-1.$$

Now Lemma 2.1 shows that this is a simplicial ordering and, hence, by Corollary 1.8, $[FS^1, FS^1]$ is a free simplicial group on a pointed simplical set. □

NOTE. 1. A similar construction works for the unit interval and, hence, by Proposition 1.6, for any wedge of circles and intervals.

2. By means of the realization functor [**2**, Appendix], [**11**], corresponding

statements can be obtained for free topological groups.

Now Theorem 2.2 shows that $[FS^1, FS^1] \cong FX$ for some pointed simplicial set X. We proceed to describe this set. It is clear that

$$S_n = \{x_0^{k_0}x_1^{k_1}\cdots x_{n-1}^{k_{n-1}} \in (FS^1)_n : x_i \in (S^1)_n,$$
$$k_i \in \mathbf{Z}, 0 \leqslant i \leqslant n-1\}$$

is the simplicial Schreier system for the subgroup $[FS^1, FS^1]$ in FS^1. In $(FS^1)_n$, write

$$w(j; k_0, \ldots, k_{n-1}) = x_0^{k_0}\cdots x_{n-1}^{k_{n-1}}x_j\left(x_0^{k_0}\cdots x_j^{k_j+1}\cdots x_{n-1}^{k_{n-1}}\right)^{-1},$$
$$0 \leqslant j \leqslant n-1.$$

Then we have

Proposition 2.3. (i) *$w(j; k_0, \ldots, k_{n-1}) \neq e$ if and only if there exists $k_i \neq 0$ for some $i > j$.*

(ii)

$$d_i w(j; k_0, \ldots, k_{n-1}) = \begin{cases} w(j; k_0, \ldots, k_i + k_{i+1}, \ldots, k_{n-1}) & \text{if } i > j \text{ and } i \neq n, \\ w(j-1; k_0, \ldots, k_i + k_{i+1}, \ldots, k_{n-1}) & \text{if } i \leqslant j \text{ and } i \neq 0, n, \end{cases}$$

$$s_i w(j; k_0, \ldots, k_{n-1}) = \begin{cases} w(j; k_0, \ldots, k_{i-1}, 0, k_i, \ldots, k_{n-1}) & \text{if } i > j, \\ w(j+1; k_0, \ldots, k_{i-1}, 0, k_i, \ldots, k_{n-1}) & \text{if } i \leqslant j, \end{cases}$$

while

$$d_0 w(j; k_0, \ldots, k_{n-1}) = \begin{cases} w(j-1; k_1, \ldots, k_{n-1}) & \text{if } j \neq 0, \\ e & \text{if } j = 0, \end{cases}$$

and

$$d_n w(j; k_0, \ldots, k_{n-1}) = w(j; k_0, \ldots, k_{n-2}).$$

Proof. Follows easily from Lemma 2.1.

The simplicial Nielsen-Schreier Theorem, Theorem 1.3 and Proposition 2.3 show that the set

$$X_n = \{w(j; k_0, \ldots, k_{n-1}) \in (FS^1)_n : \exists k_i \neq 0$$
$$\text{for some } i > j, k_i \in \mathbf{Z}, 0 \leqslant j \leqslant n-2\}$$

is a free basis for $[(FS^1)_n, (FS^1)_n]$ for each $n \geqslant 0$ so that $[FS^1, FS^1]$ is isomorphic to the simplicial group FX. We claim that X is of the same homotopy type as $S^1 \wedge AS^1$ and, hence, of S^2, since AS^1 is a $K(\mathbf{Z}, 1)$. To prove this we define $p: S^1 \wedge AS^1 \to X$ as follows:

$$p: x_j \wedge x_0^{k_0}\cdots x_{n-1}^{k_{n-1}} \mapsto w(j; k_0, \ldots, k_{n-1})$$

where $x_j \in (S^1)_n$, $0 \leqslant j \leqslant n-1$, and $k_i \in \mathbf{Z}$. It is easy to check that p is a simplicial map which identifies X with $S^1 \wedge AS^1/T$ where $T = p^{-1}(e)$, that is

$$T_n = \left\{x_j \wedge x_0^{k_0} \cdots x_j^{k_j}: x_j \in (S^1)_n, k_i \in \mathbf{Z}, 0 \leqslant j \leqslant n-1\right\}.$$

PROPOSITION 2.4. *T is contractible.*

PROOF. Let $h_s: T_n \to T_{n+1}$, $0 \leqslant s \leqslant n$, be defined by

$$h_s\left(x_j \wedge x_0^{k_0} \cdots x_j^{k_j}\right) = \begin{cases} x_s \wedge x_0^{k_0} \cdots x_j^{k_j} & \text{if } s > j, \\ x_{j+1} \wedge x_0^{k_0} \cdots x_{s-1}^{k_{s-1}} x_{s+1}^{k_s} x_{s+2}^{k_{s+1}} \cdots x_{j+1}^{k_j} & \text{if } s \leqslant j. \end{cases}$$

It can be checked easily, but tediously [**5**], that h is a contracting homotopy for T, as defined by the relations (i)–(iii) of [**11**, Definition 5.1]. □

Now it follows from the cofibration $T \overset{i}{\hookrightarrow} S^1 \wedge AS^1 \overset{p}{\hookrightarrow} X$ that p is a homotopy equivalence.

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