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MAXIMAL SEPARABLE SUBFIELDS

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ABSTRACT. If L/K is a finitely generated separable field extension of characteristic $p \neq 0$ and M is an intermediate field such that L/M is inseparable, it is proved there exist subfields S of M maximal with respect to the property that L/S is separable. These maximal separable subfields, denoted S-subfields for L/M, are characterized in two ways.

(1) Let L/S be a separable field extension. Then S is a S-subfield for L/M if and only if $S(L^p) \supseteq M$ and S is algebraically closed in M.

(2) If L/S is separable, S is a S-subfield for L/M if and only if the inseparability of L/M is equal to the transcendence degree of M/S.

A S-subfield for L/M is constructed using a maximal subset of a relative p-basis for M/K which remains p-independent in L. It is proved that there is a unique S-subfield for L/M if and only if S/K is algebraic for some S.

Throughout this paper L/K will be a finitely generated separable field extension of characteristic $p \neq 0$, and M will be an intermediate field such that L/M is inseparable. In this discussion we investigate subfields S of M/K which are maximal with respect to the property that L is separable over S. These maximal separable subfields are denoted S-subfields for L/M.

First we prove the existence of S-subfields for L/M and then illustrate one possible subfield by construction, using a subset of a relative p-basis for M/K which is maximal with respect to the property that the subset remains p-independent in L. Characterizations of S-subfields for L/M are given by two theorems. Theorem 7 states if L/S is separable, S is a S-subfield for L/M if and only if $S(L^p) \supseteq M$ and S is algebraically closed in M. We note that M will be a regular extension of any S-subfield for L/M. Theorem 8 states if L/S is regular, S is a S-subfield for L/M if and only if the inseparability of L/M is equal to the transcendence degree of M/S. From this result we obtain that all S-subfields for L/M have the same transcendence degree over K. We then determine properties of S-subfields for L/M in relation to some of the intermediate fields of L/K. Beginning with S/K, a finitely generated separable extension, and M, a regular extension of S, we construct a L such that S is a S-subfield for L/M. Finally, we show in Theorem 14 that there will exist a unique S-subfield for L/M if and only if S/K is algebraic for some S, where S is a S-subfield for L/M. A field L is said to be *separable* over a subfield K if and only if L^p (the field of pth powers of L) and K are linearly disjoint over K^p. The *inseparability* of L/M, denoted

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insep L/M, is equal to the number of elements in a relative p-basis for L/M minus the number of elements in a transcendency basis for L/M.

DEFINITION 1. If L/K is separable and finitely generated and if L is inseparable over an intermediate field M, then a subfield S of M containing K is a *S-subfield for L/M* if and only if S is a maximal subfield of M with respect to the property that L/S is separable.

The following known result is used to prove the existence of S-subfields for L/M.

LEMMA 2. *If L/K is a finitely generated field extension, then any intermediate field is finitely generated over K.*

THEOREM 3. *If L/K is finitely generated and separable, and L is inseparable over an intermediate field M, then there exist S-subfields for L/M.*

PROOF. Let $\mathbb{S}$ be the collection S_α of subfields of M/K such that L is separable over each S_α. Let set inclusion be a partial ordering on $\mathbb{S}$. Since L/K is separable, $K \in \mathbb{S}$ and $\mathbb{S} \neq \varnothing$. Let $\mathcal{C}$ be a chain in $\mathbb{S}$ and let $E = \bigcup_{S_\alpha \in \mathcal{C}} S_\alpha$, which is clearly an upper bound of $\mathcal{C}$. Since $\mathcal{C}$ is a totally ordered subcollection of S, E must be a subfield of M/K. By Lemma 2, E is finitely generated over K. Let $E = K(\beta_1, \beta_2, \ldots, \beta_r)$. By the definition of E, $\beta_1, \beta_2, \ldots, \beta_r \in S_\alpha$ for some α. Hence, $K(\beta_1, \beta_2, \ldots, \beta_r) = S_\alpha$ and L/E is separable. This implies $E \in \mathbb{S}$. By Zorn's Lemma there exists a maximal element of $\mathbb{S}$. We conclude there exist S-subfields for L/M.

In general there does not exist a unique S-subfield for L/M. We show in Theorem 14 precisely when there will be a unique maximal subfield. A S-subfield for L/M can be constructed using a subset of a relative p-basis for M/K which remains p-independent in L. Let B be a maximal subset of a relative p-basis for M/K which remains p-independent in L. Then $L/K(B)$ is separable [**5**, p. 378].

PROPOSITION 4. *Let L/S be separable where $L \supset M \supset S \supset K(B) \supset K$. If B is a maximal subset of a relative p-basis for M/K which remains p-independent in L, then S must be an algebraic extension of $K(B)$.*

PROOF. Suppose S is a transcendental extension of $K(B)$. We noted above that $L/K(B)$ is separable; hence, $S/K(B)$ is separable and has a separating transcendency basis [**5**, Theorem 3, p.373]. Let T be a separating transcendency basis for S over $K(B)$. The basis $T \neq \varnothing$ since $S/K(B)$ is transcendental. Let $B \cup T$ be a relative p-basis for S/K. Since L/S is separable, $B \cup T$ remains relatively p-independent in L/K [**5**, p. 378]. This implies $B \cup T$ is relatively p-independent in M/K and contradicts the hypothesis that B is a maximal subset of a relative p-basis for M/K which remains p-independent in L. We conclude S is an algebraic extension of $K(B)$.

Therefore, any extension S of $K(B)$ contained in M with the property that L/S is separable must be algebraic. A S-subfield for L/M will be the largest algebraic extension of $K(B)$ in M. Hence, the algebraic closure of $K(B)$ in M is a S-subfield for L/M.

PROPOSITION 5. *The field extension M/S is not algebraic.*

PROOF. Since L/S is separable, M/S is separable. If M/S is algebraic, then L/M is separable [**1**, p. 8]. This contradicts the assumption that L/M is inseparable. Hence, M/S is not algebraic.

PROPOSITION 6. *The field S is algebraically closed in M.*

PROOF. Let $s \in M$ and s be algebraic over S. Then $L/S(s)$ must be separable since L/S is separable and $S(s)/S$ is algebraic. Since S is a S-subfield for L/M, S is a maximal subfield of M with respect to the separability of L/S; hence s must belong to S, and we conclude that S is algebraically closed in M.

The next two theorems are useful characterizations of S.

THEOREM 7. *If L/S is separable, S is a S-subfield for L/M if and only if $S(L^p) \supseteq M$ and S is algebraically closed in M.*

PROOF. Suppose S is a S-subfield for L/M. We have shown that S is algebraically closed in M. Let $\theta \in M$ but $\theta \notin S$. The element θ must be transcendental over S; hence, $\{\theta\}$ is a relative p-basis for $S(\theta)/S$. Since S is maximal in M, $L/S(\theta)$ cannot be separable. Therefore, θ is not relatively p-independent in L/S. Let B be a p-basis for S. Then $B \cup \{\theta\}$ is a p-basis for $S(\theta)$. Since B remains p-independent in L by the separability of L/S and since $B \cup \{\theta\}$ does not remain p-independent in L, $\theta \in L^p(B) \subseteq S(L^p)$. Since θ was arbitrary, $M \subseteq S(L^p)$. Conversely, suppose S is algebraically closed in M and $M \subseteq S(L^p)$. Consider an extension S' of S contained in M. Let B' be a relative p-basis for S'/S. The basis $B' \neq \emptyset$ since S'/S must be transcendental. From the hypothesis and since $B' \subseteq M$, $S(L^p) = S(L^p, B')$ which implies that B' does not remain relatively p-independent in L, and L/S' cannot be separable. Hence, S is a maximal subfield of M such that L/S is separable.

Let $p_{L/M}$, $p_{L/S}$, and $p_{M/S}$ represent relative p-bases for L/M, L/S, and M/S, respectively.

THEOREM 8. *Let L/S be separable. Then S is a S-subfield for L/M if and only if* $\operatorname{tr\,d} M/S = \operatorname{insep} L/M$ *and S is algebraically closed in M.*

PROOF. Suppose S is a S-subfield for L/M. Since $p_{L/M}$ and $p_{L/S}$ are relative p-bases for L/M and L/S respectively, by definition $L = M(L^p, p_{L/M})$ and $L = S(L^p, p_{L/S})$. By Theorem 7, $M \subseteq S(L^p)$, therefore $L = S(L^p, p_{L/M})$. Hence, a relative p-basis for L/M will be a relative p-basis for L/S and $|p_{L/M}| = |p_{L/S}|$.

We note that

$$\operatorname{tr\,d} L/S = \operatorname{tr\,d} L/M + \operatorname{tr\,d} M/S. \tag{8.a}$$

By the separability of L/S there exists a separating transcendency basis for L/S [**5**, Theorem 3, p. 373] and $\operatorname{tr\,d} L/S = |p_{L/S}|$ [**5**, Lemma 3, p. 382]. From the definition of inseparability we obtain the equality

$$\operatorname{tr\,d} L/M + \operatorname{insep} L/M = |p_{L/M}|;$$

therefore,

$$\operatorname{tr\,d} L/M + \operatorname{insep} L/M = |p_{L/S}|. \tag{8.b}$$

Equating (8.a) and (8.b), we obtain

$$\operatorname{tr\,d} L/M + \operatorname{tr\,d} M/S = \operatorname{tr\,d} L/M + \operatorname{insep} L/M.$$

Hence,

$$\operatorname{tr\,d} M/S = \operatorname{insep} L/M.$$

By Proposition 6, S is algebraically closed in M. Conversely, suppose $\operatorname{tr\,d} L/S = \operatorname{insep} L/M$ and S is algebraically closed in M. If S is not a maximal subfield of M such that L/S is separable, then S is contained in some S' which is a S-subfield for L/M. By the first part of this theorem $\operatorname{tr\,d} M/S' = \operatorname{insep} L/M$. By hypothesis $\operatorname{tr\,d} M/S' = \operatorname{tr\,d} M/S$; therefore, S' must be algebraic over S. Since S is algebraically closed in M, $S = S'$, and S is a S-subfield for L/M.

COROLLARY 9. *All S-subfields for L/M have the same transendence degree over K.*

PROOF. This follows easily from the fact that $\operatorname{insep} L/M = \operatorname{tr\,d} M/S$ for any S-subfield for L/M.

We note that a S-subfield for L/M is algebraically closed in M, and M/S is a separable extension. Hence, M is a regular extension of each S-subfield for L/M.

Assume that L/M is an algebraic extension and L is inseparable over M. There exists a unique intermediate field M' such that M'/M is separable and L/M' is purely inseparable [**2**, pp. 46–47].

THEOREM 10. *If L/M is algebraic and M' is the intermediate field such that L/M' is purely inseparable and M'/M is separable, then the algebraic closure in M' of a S-subfield for L/M will be a S-subfield for L/M'. Moreover, if S' is a S-subfield for L/M' and $S'/(S' \cap M)$ is algebraic, then $S' \cap M$ is a S-subfield for L/M.*

PROOF.

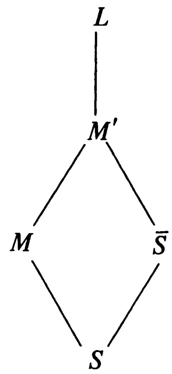

FIGURE (i)

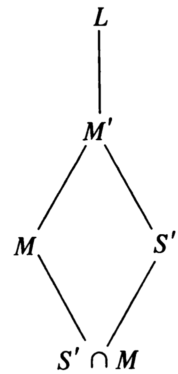

FIGURE (ii)

(i) Suppose $\bar{S}$ is the algebraic closure in M' of S, where S is a S-subfield for L/M. Since M'/M is algebraic and $\bar{S}$ is algebraically closed in M' with S algebraically closed in M, $\operatorname{tr\,d} M'/\bar{S} = \operatorname{tr\,d} M/S$. By Theorem 8, $\operatorname{tr\,d} M/S = \operatorname{insep} L/M$ and since M'/M is separable, $\operatorname{insep} L/M = \operatorname{insep} L/M'$ [**3**, p. 112]. Hence, $\operatorname{tr\,d} M'/\bar{S} = \operatorname{insep} L/M'$. Since L/S is separable and $\bar{S}/S$ is algebraic, $L/\bar{S}$ is separable. By Theorem 8, $\bar{S}$ is a S-subfield for L/M.

(ii) Suppose S' is a S-subfield for L/M' and S' is algebraic over $S' \cap M$. To show $S' \cap M$ is algebraically closed in M, let $\alpha \in M$ and α be an algebraic element over $S' \cap M$. Since $\alpha \in M'$ and S' is algebraically closed in M', $\alpha \in S'$. Hence, $\alpha \in S' \cap M$. Since M'/M is separable, $\operatorname{insep} L/M = \operatorname{insep} L/M'$. The extension M'/M is algebraic and $S'/(S' \cap M)$ is algebraic; hence, $\operatorname{tr\,d} M'/S' = \operatorname{tr\,d} M/(S' \cap M)$. We see that $\operatorname{tr\,d} M/(S' \cap M) = \operatorname{insep} L/M' = \operatorname{insep} L/M$ since S' is a S-subfield for L/M' [Theorem 8]. Finally, we must show that $L/(S' \cap M)$ is separable. Since M'/S' is separable and M'/M is separable, M' is separable over $S' \cap M$. The field S' is a subfield of M'; hence, S' is separable over $S' \cap M$. Then since L/S' is separable and $S'/(S' \cap M)$ is separable, it follows that $L/(S' \cap M)$ is separable. We conclude that $S' \cap M$ is a S-subfield for L/M.

More generally, let L/M be an inseparable extension and S be a S-subfield for L/M.

PROPOSITION 11. *If E is an intermediate field of M/S where S is a S-subfield for L/M, then S is a S-subfield for L/E.*

PROOF. This follows easily from Theorem 7.

Proposition 11 implies that a relative p-basis for L/M will be a relative p-basis for L/E and will also be a relative p-basis for L/S [Theorem 7].

COROLLARY 12. *If E is an intermediate field of M/S where S is a S-subfield for L/M and M/E is algebraic, then* $\operatorname{insep} L/M = \operatorname{insep} L/E$.

PROOF. Since S is a S-subfield for L/E, insep $L/E = \text{tr d } E/S = \text{tr d } M/S =$ insep L/M by Theorem 8.

Suppose S is a separable extension of K and M is a regular extension of S. A field L can be constructed such that L/M is inseparable and S is a S-subfield for L/M.

THEOREM 13. *If S/K is a separable extension and M/S is regular, there exists an inseparable extension L/M such that $S(L^p) \supseteq M$ and L/S is separable.*

PROOF. The field M is finitely generated over S. Let $M = S(t_1, t_2, \ldots, t_s, \alpha)$ where $\{t_1, t_2, \ldots, t_s\}$ is a separating transcendency basis for M/S. Let $L = M(t_1^{1/p}, t_2^{1/p}, \ldots, t_s^{1/p})$. Clearly, $S(L^p) \supseteq M$. By hypothesis M/S is separable and

$$M\left(t_1^{1/p}, t_2^{1/p}, \ldots, t_s^{1/p}\right) = S\left(t_1^{1/p}, t_2^{1/p}, \ldots, t_s^{1/p}, \alpha\right).$$

Since $t_1, t_2, \ldots, t_s$ are algebraically independent over S, their pth roots are also algebraically independent over S. The element α is separable over $S(t_1, t_2, \ldots, t_s)$ by construction. Hence L/S is separable.

Clearly, the L constructed in Theorem 13 is a minimal extension of M/S such that S is a S-subfield for L/M.

THEOREM 14. *There will exist a unique S-subfield for L/M if and only if S/K is algebraic for some S, where S is a S-subfield for L/M.*

PROOF. Assume S/K is algebraic for some S. Suppose there exists another subfield S' of M which is a S-subfield for L/M. The field S' must be algebraically closed in M. If $\alpha \in S$, α is algebraic over K, hence α is algebraic over S'. Since S' is algebraically closed in M, $\alpha \in S'$. Hence, $S \subseteq S'$. But S is maximal, so we conclude $S = S'$ and S is unique.

Conversely, suppose S/K is transcendental. Then there exists a finite transcendency basis. Let $\{t_1, t_2, \ldots, t_r\}$ be such a basis. These elements form a relative p-basis for S/K; hence they are relatively p-independent in L/K by the separability of L/S [5, p. 378]. Let $M = S(\omega_1, \omega_2, \ldots, \omega_s, \beta)$ where $\{\omega_1, \omega_2, \ldots, \omega_s\}$ is a separating transcendency basis for M/S and β is separable algebraic over $S(\omega_1, \omega_2, \ldots, \omega_s)$. There is at least one ω_i since M/S cannot be algebraic. Consider the subfield $K(\omega_1 + t_1, t_2, \ldots, t_r)$, a pure transcendental extension of K. If $(\omega_1 + t_1) \in K(L^p)(t_2, t_3, \ldots, t_r)$ and $\omega_1 \in K(L^p)(t_2, t_3, \ldots, t_r)$, then $t_1 \in K(L^p)(t_2, t_3, \ldots, t_r)$. This is impossible since $(t_1, t_2, \ldots, t_r)$ is relatively p-independent in L/S. Hence, either $\omega_1 + t_1$ or ω_1 does not belong to $K(L^p)(t_2, t_3, \ldots, t_r)$. Let t_1' represent the element ω_1 or $\omega_1 + t_1$ not contained in $K(L^p)(t_1, t_2, \ldots, t_r)$. Let $S' = K(t_1', t_2, \ldots, t_r)$ and let $\bar{S}'$ be its algebraic closure in M. The set $\{t_1', t_2, \ldots, t_r\}$ is relatively p-independent in L/K and $\{t_1', t_2, \ldots, t_r\}$ is a transcendency basis for $\bar{S}'/K$, hence a relative p-basis for $\bar{S}'/K$. By [5, Lemma 3, p. 382], $L/\bar{S}'$ is separable. Since $r = \text{tr d } M/\bar{S}' =$ insep L/M, $\bar{S}'$ is a S-subfield for L/M [Theorem 8]. By the construction of S', $t_1' \notin S$; hence, $\bar{S}' \neq S$ and S is not unique.

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