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STABLE EQUIVALENCE OF UNISERIAL RINGS

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ABSTRACT. It is well known that two basic rings are Morita equivalent if and only if they are isomorphic. Here it is shown that local uniserial rings are stably equivalent in the sense of M. Auslander and I. Reiten in case they are isomorphic modulo certain powers of their radicals. In particular, two commutative local uniserial rings of Loewy length n are stably equivalent if and only if they are isomorphic modulo the $[n/2]$th power of their radicals.

In a series of papers [2]–[6], M. Auslander and I. Reiten have studied categories of modules modulo projectives and the related notion of stable equivalence of rings. We recall their terminology. For a ring Λ, mod Λ denotes the category of finitely generated left Λ-modules. For $M, N \in$ mod Λ, $P_\Lambda(M, N)$ is the subgroup of $\mathrm{Hom}_\Lambda(M, N)$ consisting of the homomorphisms f: $M \to N$ that factor through projectives. The additive category of finitely generated left Λ-modules modulo projectives is $\underline{\mathrm{mod}}\ \Lambda$. Its objects are the same as those of mod Λ, and its morphisms are the members of the factor groups $\underline{\mathrm{Hom}}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)/P_\Lambda(M, N)$. The category $\mathrm{mod}_P\ \Lambda$ is the full subcategory of mod Λ whose objects have no projective direct summands, and $\underline{\mathrm{mod}}_P\ \Lambda$ is the corresponding subcategory of $\underline{\mathrm{mod}}\ \Lambda$. If Λ is a left artinian ring then $\underline{\mathrm{mod}}_P\ \Lambda \approx \underline{\mathrm{mod}}\ \Lambda$ ($\approx$ denotes equivalence of categories) [3, p. 247]. Two rings Λ and Λ' are said to be (*left*) *stably equivalent* in case $\underline{\mathrm{mod}}\ \Lambda \approx \underline{\mathrm{mod}}\ \Lambda'$.

Stable equivalence is a relaxing of the notion of Morita equivalence. So since artinian rings are Morita equivalent iff their basic rings are isomorphic (see [1, §27]), one is led to study the relative structure of stably equivalent rings. For example, although $\mathbf{Z}_{p^2}$ and $\mathbf{Z}_p[x]/(x^2)$ are not Morita equivalent, they are stably equivalent [2]. See also [6] and [7] regarding stable equivalence of serial rings.

Here we consider stable equivalence of uniserial rings. A uniserial ring is one that is Morita equivalent to a direct sum of local uniserial rings. A ring Λ with $J = J(\Lambda)$ and Loewy length n is a local uniserial ring in case $\Lambda > J > \cdots > J^{n-1} > J^n = 0$ is a complete list of the submodules of both ${}_\Lambda\Lambda$ and Λ_Λ. The indecomposable modules over Λ are then $\Lambda/J, \ldots, \Lambda/J^{n-1}, \Lambda$, and $P_\Lambda(\Lambda/J^i, \Lambda/J^j)$ consists of those maps that factor through Λ. In what

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follows we shall write morphisms on the right so that $M \to^{f} N \to^{g} L$ composes to fg, and for a number n we shall write

$$[n/2] = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

1. THEOREM. *Let Λ and Λ' be local uniserial rings with radicals J and J' and Loewy lengths n and n', respectively.*

(1) *If Λ is stably equivalent to Λ' then $n = n'$ and $\Lambda/J^{[n/2]} \cong \Lambda'/J'^{[n/2]}$.*

(2) *If $n = n'$ and $\Lambda/J^{n-1} \cong \Lambda'/J'^{n-1}$ then Λ is stably equivalent to Λ'.*

PROOF. (1) First note that in any sequence of Λ-maps

$$\Lambda/J^{[n/2]} \to \Lambda \to \Lambda/J^{[n/2]}$$

the image of the first must be contained in the kernel of the second; then observe that the endomorphism ring of any indecomposable Λ'-module is isomorphic to a factor ring of Λ'. Thus, since M is indecomposable in $\underline{\text{mod}}\ \Lambda'$ iff ${}_{\Lambda'}M$ is indecomposable and not projective [3, p. 247], we see that if F: $\underline{\text{mod}}\ \Lambda \to \underline{\text{mod}}\ \Lambda'$ is an equivalence then

$$\begin{aligned} \Lambda/J^{[n/2]} &\cong \text{End}_\Lambda(\Lambda/J^{[n/2]}) \cong \underline{\text{End}}_\Lambda(\Lambda/J^{[n/2]}) \\ &\cong \underline{\text{End}}_{\Lambda'}\big(F(\Lambda/J^{[n/2]})\big) \cong \Lambda'/J'^{l} \end{aligned}$$

where, of necessity, $l = [n/2]$.

(2) If Λ is local and uniserial of length n then $\text{mod}_P\ \Lambda \cong \text{mod}\ \Lambda/J^{n-1}$. Thus if $\Lambda/J^{n-1} \cong \Lambda'/J'^{n-1}$, we have

$$\underline{\text{mod}}\ \Lambda \approx \underline{\text{mod}}_P\ \Lambda \approx \underline{\text{mod}}\ \Lambda/J^{n-1} \approx \underline{\text{mod}}\ \Lambda'/J'^{n-1} \approx \underline{\text{mod}}\ \Lambda'.$$

We shall see that this theorem is "best possible" in the sense that the converse of each of its parts is false. It will be convenient, however, to examine the commutative case first. For commutative rings the situation is a bit more complicated, so we require the following two lemmas.

2. LEMMA. *Let Λ and Λ' be two rings of finite module type. Let U and U' be direct sums of one copy of each indecomposable left module over Λ and Λ', respectively. Then Λ is stably equivalent to Λ' if and only if $\underline{\text{End}}_\Lambda(U) \cong \underline{\text{End}}_{\Lambda'}(U')$.*

PROOF. An equivalence F: $\underline{\text{mod}}\ \Lambda \to \underline{\text{mod}}\ \Lambda'$ restricts to a ring isomorphism F: $\underline{\text{End}}_\Lambda(U) \to \underline{\text{End}}_{\Lambda'}(F(U))$. But in $\underline{\text{mod}}\ \Lambda'$ we must have $F(U) \cong U'$.

Conversely, an isomorphism Φ: $\underline{\text{End}}_\Lambda(U) \to \underline{\text{End}}_{\Lambda'}(U)$ can be extended to an equivalence F: $\underline{\text{mod}}\ \Lambda \to \underline{\text{mod}}\ \Lambda'$ because the objects in $\underline{\text{mod}}\ \Lambda$ and $\underline{\text{mod}}\ \Lambda'$ are all direct summands of direct sums of copies of U and U', respectively.

Next we calculate $\underline{\text{End}}_\Lambda(U)$ for a local uniserial ring Λ.

3. LEMMA. *Let Λ be a local uniserial ring with radical J and Loewy length n. Let $U = \Lambda/J \oplus \cdots \oplus \Lambda/J^{n-1} \oplus \Lambda$. If, for $i, j \in \{1, \ldots, n\}$, $A = [A_{ij}]$ where*

$$A_{ij} = \begin{cases} \Lambda, & i \geqslant j, \\ J^{j-i}, & i < j, \end{cases}$$

and $B = [B_{ij}]$ *where*

$$B_{ij} = \begin{cases} J^j, & i + j \leqslant n, \\ J^{n-i}, & i + j > n, \end{cases}$$

then A *is a subring of the ring of* $n \times n$ *matrices over* Λ, *and* B *is an ideal of* A *such that*

$$\underline{\mathrm{End}}_\Lambda(U) \cong A/B.$$

PROOF. The ring $\mathrm{End}_\Lambda(U)$ is canonically isomorphic to the ring of $n \times n$ matrices $[\mathrm{Hom}_\Lambda(\Lambda/J^i, \Lambda/J^j)]$, and under this isomorphism, $P_\Lambda(U)$ corresponds to $[P_\Lambda(\Lambda/J^i, \Lambda/J^j)]$. For $i, j \in \{1, \ldots, n\}$, let

$$\theta_{ij}: A_{ij} \to \mathrm{Hom}_\Lambda(\Lambda/J^i, \Lambda/J^j)$$

via

$$\theta_{ij}(a_{ij}): \lambda + J^i \mapsto \lambda a_{ij} + J^j.$$

Then the θ_{ij} are surjective Λ-homomorphisms such that

$$\begin{aligned}(\lambda + J^i)[\theta_{ik}(a_{ik})][\theta_{kj}(d_{kj})] &= (\lambda a_{ik} + J^k)[\theta_{kj}(d_{kj})] \\ &= \lambda a_{ik} d_{kj} + J^j = (\lambda + J^1)[\theta_{ij}(a_{ik} d_{kj})]\end{aligned}$$

for all $a_{ik} \in A_{ik}$, $d_{kj} \in A_{kj}$. Thu s the θ_{ij} induce a surjective ring homomorphism $\Theta: A \to \underline{\mathrm{End}}_\Lambda(U)$. (It is easily shown that A is a subring of the matrix ring $M_n(\Lambda)$.) Regarding $P_\Lambda(U)$, we see that

$$\begin{aligned}P(\Lambda/J^i, \Lambda/J^j) &= \mathrm{Hom}(\Lambda/J^i, \Lambda)\mathrm{Hom}(\Lambda, \Lambda/J^j) \\ &= \theta_{in}(A_{in})\theta_{nj}(A_{nj}) = \theta_{ij}(A_{in}A_{nj}).\end{aligned}$$

Now observe that

$$A_{in}A_{nj} = J^{n-i}$$

and let $C_{ij} = \mathrm{Ker}\ \theta_{ij}$. Then

$$C_{ij} = J^j \quad \text{and} \quad \mathrm{Ker}\ \Theta = [C_{ij}],$$

and we have

$$\Theta^{-1}(P_\Lambda(U)) = [A_{in}A_{nj}] + [C_{ij}] = [B_{ij}].$$

Thus $B = [B_{ij}]$ is an ideal in A and $\underline{\mathrm{End}}_\Lambda(U) \cong A/B$.

Now we can prove the converse of part (1) of Theorem 1 for commutative rings. That is, commutative uniserial rings are stably equivalent if and only if their "top halves" are isomorphic.

4. THEOREM. *Let* Λ *and* Λ' *be commutative local uniserial rings with Loewy lengths* n *and* n' *and radicals* J *and* J', *respectively. Then* Λ *is stably equivalent to* Λ' *if and only if* $n = n'$ *and* $\Lambda/J^{[n/2]} \cong \Lambda'/J'^{[n/2]}$.

PROOF. The condition is necessary by Theorem 1.

For sufficiency, suppose that Λ and Λ' are commutative of length n and that

$$\varphi\colon \Lambda/J^m \to \Lambda'/J'^m \qquad (m = [n/2])$$

is a ring isomorphism. Let $A = [A_{ij}]$ and $B = [B_{ij}]$ be as in Lemma 3, and define $A' = [A'_{ij}]$ and $B' = [B'_{ij}]$ analogously for Λ'. From the defining inequalities and the equality $m = [n/2]$, one can quickly check that

$$J^m A_{ij} \subseteq B_{ij} \quad \text{and} \quad J'^m A'_{ij} \subseteq B'_{ij},$$

so that A/B and A'/B' are algebras over Λ/J^m and Λ'/J'^m, respectively. Now choose $x \in J$ and $x' \in J'$ such that

$$\Lambda x = J, \quad \Lambda' x' = J' \quad \text{and} \quad \varphi(x + J^m) = x' + J'^m.$$

For $i, j \in \{1, \ldots, n\}$, let

$$t_{ij} = \begin{cases} 0, & i \geqslant j, \\ j - i, & i < j, \end{cases}$$

and let

$$x_{ij} = e_{ij}(x^{t_{ij}}) + B, \qquad x'_{ij} = e_{ij}(x'^{t_{ij}}) + B',$$

where $e_{ij}(a)$ is the $n \times n$ matrix whose ijth entry is a and whose other entries are all zero. Then A/B and A'/B' are spanned by $\{x_{ij}\}$ and $\{x'_{ij}\}$ over Λ/J^m and Λ'/J'^m, respectively, and

$$\Phi\colon (\lambda_{ij} + J^m)x_{ij} \mapsto \varphi(\lambda_{ij} + J^m)x'_{ij}$$

clearly defines a bijective additive mapping

$$\Phi\colon A/B \to A'/B'.$$

But we see that Φ is in fact a ring isomorphism by first observing that

$$t_{ik} + t_{kj} \geqslant t_{ij}$$

(which seems to require checking four cases) and then calculating

$$\begin{aligned} \Phi(x_{ik}x_{kj}) &= \Phi\big(e_{ij}(x^{t_{ik}}x^{t_{kj}}) + B\big) \\ &= \Phi\big((x^{t_{ik}+t_{kj}-t_{ij}} + J^m)x_{ij}\big) \\ &= (x'^{t_{ik}+t_{kj}-t_{ij}} + J'^m)x'_{ij} \\ &= x'_{ik}x'_{kj} = \Phi(x_{ik})\Phi(x_{kj}). \end{aligned}$$

Thus $A/B \cong A'/B'$, and by the lemmas the proof is complete.

It is known [2] that uniserial rings of length 2 are stably equivalent provided they are isomorphic modulo the radical. For commutative rings

5. COROLLARY. *If Λ and Λ' are commutative uniserial rings of Loewy length 3, then Λ is stably equivalent to Λ' if and only if $\Lambda/J(\Lambda) \cong \Lambda'/J(\Lambda')$.*

Next, aided by the last results, we present several examples which show that the thoerems cannot be sharpened. These examples may also shed some light on the essence of stable equivalence.

6. EXAMPLES. (1) Let $\mathbf{C}[x;\ ^{-}]$ denote the ring of complex polynomials with multiplication twisted by conjugation, i.e., $xc = \bar{c}x$. Let $\Lambda = \mathbf{C}[x;\ ^{-}]/(x^3)$ and $\Lambda' = \mathbf{C}[x]/(x^3)$. Then upon constructing A/B and A'/B' for Λ and Λ' (see Lemma 3), one observes that the center of A/B is isomorphic to the real numbers while that of A'/B' is isomorphic to the complex numbers. Thus by Lemmas 2 and 3, Λ is not stably equivalent to Λ'. But because $n = n' = 3$, $[n/2] = 1 = n - 2$, and $\Lambda/J \cong \mathbf{C} \cong \Lambda'/J'$, we see that the converse of part (1) of Theorem 1 is false, the condition of part (2) cannot be relaxed, and commutativity is necessary in Corollary 5.

(2) By Theorem 4, $\mathbf{Z}_{p^n}$ is stably equivalent to $\mathbf{Z}_p[x]/(x^n)$ if and only if $n = 1, 2$ or 3. This shows in particular that the converse of part (2) of Theorem 1 is false.

(3) Let $\Lambda = \mathbf{Z}_4[x]/(x^2 - 2)$. Then, letting $\bar{f} = f + (x^2 - 2)$, we have $J = J(\Lambda) = \Lambda\bar{x}$, $J^2 = \Lambda\bar{2}$, $J^3 = \Lambda\overline{2x}$, and $J^4 = 0$. Thus if $\Lambda' = \mathbf{Z}_2[x]/(x^4)$ then $\Lambda/J^2 \cong \Lambda'/J'^2$, so Λ is stably equivalent to Λ' by Theorem 4. But $\operatorname{Char}(\Lambda/J^3) = 4$ and $\operatorname{Char}(\Lambda'/J'^3) = 2$, so even for $n \geqslant 4$, stable equivalence does not imply isomorphism any more than "halfway down."

(4) As a final example, we note that the rings $\Lambda = \mathbf{Z}_8[x]/(x^3 - 2)$ and $\Lambda' = \mathbf{Z}_2[x]/(x^9)$ are isomorphic modulo the cubes of their radicals, but are not stably equivalent.

Theorem 4 leads us to consider stable equivalence for discrete valuation domains. The answer is just about what one would expect. We denote the completion of a discrete valuation domain Δ by $\hat{\Delta}$. Then we conclude with

7. PROPOSITION. *Let Δ and Δ' be discrete valuation domains. Then Δ is stably equivalent to Δ' if and only if $\hat{\Delta} \cong \hat{\Delta}'$.*

PROOF. Since $\operatorname{mod}_P \Delta$ consists of the Δ-modules of finite length and only the zero map between two such modules factors through a projective, Δ is stably equivalent to Δ' if and only if there is an equivalence F: $\operatorname{mod}_P \Delta \to \operatorname{mod}_P \Delta'$. But the modules over Δ and $\hat{\Delta}$ of finite length are the same [**8**, p. 37], so isomorphic completions yield such an equivalence. Conversely, given F, let $M_i = \Delta/J^i$ and let ι_i: $M_i \to M_{i+1}$ and π_i: $M_{i+1} \to M_i$ be a monomorphism and an epimorphism, respectively, $i = 1, 2, \ldots$. Then if η_i and η_i' are the canonical surjective ring homomorphisms, we have commuting diagrams

$$\begin{tikzcd}
\Delta/J^{i+1} \arrow[r, "\rho"] \arrow[d, "\eta_i"'] & \operatorname{End}_\Delta(M_{i+1}) \arrow[r, "F"] \arrow[d, "{(\iota_i,\, \pi_i)}"'] & \operatorname{End}_{\Delta'}(F(M_{i+1})) \arrow[r, "\rho'^{-1}"] \arrow[d, "{(F(\iota_i),\, F(\pi_i))}"'] & \Delta'/J'^{i+1} \arrow[d, "\eta_i'"'] \\
\Delta/J^i \arrow[r, "\rho"'] & \operatorname{End}_\Delta(M_i) \arrow[r, "F"'] & \operatorname{End}_{\Delta'}(F(M_i)) \arrow[r, "\rho'^{-1}"'] & \Delta'/J'^i
\end{tikzcd}$$

So that (see [**8**, p. 37])

$$\hat{\Delta}\varprojlim(\Delta/J^i) \cong \varprojlim(\Delta'/J'^i) = \hat{\Delta}'.$$

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