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ELEMENTARY SURGERY MANIFOLDS AND THE ELEMENTARY IDEALS

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ABSTRACT. We prove the following: If M^3 is a closed 3-manifold obtained by elementary surgery on a knot K in S^3 and $H_1(M^3)$ is a nontrivial cyclic group, then the first elementary ideal $\pi_1(M^3)$ in the integral group ring of $H_1(M^3)$ is the principal ideal generated by the polynomial of K.

In this paper we study the 3-manifolds which are obtained by elementary surgery along a knot in S^3 and which are not homology spheres. This allows us to use the free calculus. Our main result is the following: If M^3 is a closed 3-manifold obtained by elementary surgery along a knot K in S^3 and $H_1(M^3)$ is a nontrivial cyclic group C, then the first elementary ideal of $\pi_1(M^3)$ in the integral group ring of C is the principal ideal generated by the first knot polynomial of K.

We will use the notation of the free calculus as developed in Chapter VII of [**2**]. For $n > 0$, we will use Z_n to denote the cyclic group of order n, and Z_0 will be the infinite cyclic group. The generator of Z_n will be t. $J(Z_n)$ will denote the integral group ring of Z_n, thus the elements of $J(Z_n)$, for $n > 0$, are finite sums $a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$ where each a_i is an integer. We will use τ to denote the trivializer of $J(Z_n)$, that is, $\tau: J(Z_n) \to Z$ is defined by $\tau(t) = 1$, so that $\tau(f(t)) = f(1)$. For $n > 0$, Λ_n will denote the element $\Sigma_{i=0}^{n-1} t^i$ of $J(Z_n)$ and we let $\Lambda_0 = 0$. Note that $t^k \Lambda_n = \Lambda_n$ for all k, hence, for any $f(t) \in J(Z_n)$,

$$f(t)\Lambda_n = \tau(f(t))\Lambda_n = f(1)\Lambda_n.$$

If $f(t) \in J(Z_n)$, $(f(t))$ will denote the principal ideal generated by $f(t)$.

We use $\langle x_1, \ldots, x_k | R_1, \ldots, R_m \rangle$ to denote a group given by generators and relations. Let $G = \langle x_1, \ldots, x_k | R_1, \ldots, R_m \rangle$. Suppose the abelianization of G is Z_n. We use ϕ to denote the corresponding homomorphism of the free group $\langle x_1, \ldots, x_k | \rangle$ onto Z_n. Thus the Alexander matrix of G has $\phi(\partial R_i / \partial x_j)$ as the entry in the ith row and the jth column. We let $E_n(G)$ denote the nth elementary ideal of G as defined in [**2**, p. 101]. If K is a knot in S^3, we let $E_n(\pi_1(S^3 - K)) = E_n(K)$. For a 3-manifold M^3, we let $E_n(\pi_1(M^3)) = E_n(M^3)$.

Let K be a knot in S^3 and let N be a solid tubular neighborhood of K. Then N is a solid torus. Let (m, l) be a meridian-longitude pair for N. (This

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means that m and l are simple closed curves on Bd N, m bounds a disk in N but not on Bd N, and l is a homology generator of N.) In addition, assume that l is homologically trivial in $\mathrm{Cl}(S^3 - N)$. The elementary surgery manifold $M^3(K; n, s)$ is constructed as follows: remove Int N from S^3 and sew in a new solid torus T so that a meridian of T is sewn to a curve C which is homologous to $nm + sl$ on Bd N. If $n \neq 0$, n and s must be relatively prime and if $n = 0$, we must have $s = \pm 1$. Note that $H_1(M^3(K; n, s)) = Z_{|n|}$; hence in this paper we restrict our attention to the case $|n| \neq 1$. Note also that $\pi_1(M^3(K; n, s))$ may be obtained by adding to $\pi_1(S^3 - K)$ a relation which trivializes the element of $\pi_1(S^3 - K)$ corresponding to the curve C. Since $M^3(K; n, s)$ and $M^3(K; -n, -s)$ are homeomorphic, we will assume throughout that n is nonnegative. If J is a simple closed curve in a space X, we will use J to denote the simple closed curve itself, the corresponding element of $\pi_1(X)$ and the corresponding element of $H_1(X)$. Finally, if S is a set of integers, we will use GCD S to denote the greatest common divisor of the elements in S. The proofs of Theorems 1 and 2 given here were suggested to the author by the referee and represent a substantial improvement over the original proofs.

THEOREM 1. *The Alexander matrix of $\pi_1(M^3(K; n, s))$ is obtained from the matrix of $\pi_1(S^3 - K)$ by adjoining a new row with one entry Λ_n and the rest zeros. In addition, the other entries in the colmun containing Λ_n are all zeros.*

We should note here that when we say that the Alexander matrix is a certain matrix, we always mean up to equivalence as defined in [2, p. 101].

PROOF OF THEOREM 1. Let $G = \pi_1(S^3 - K)$ and let G' denote the commutator subgroup of G. G has a presentation $\langle a, x_1, \ldots, x_m | R_1, \ldots, R_m \rangle$, where a is a meridian of K, $x_j \in G'$ for $1 \leqslant j \leqslant m$, and $\phi(\partial R_i/\partial a) = 0$ for $1 \leqslant i \leqslant m$ [3, p. 415]. Hence G has Alexander matrix $(\phi(\partial R_i/\partial x_j)|0)$. Now the matrix of $\pi_1(M^3(K; n, s))$ is obtained by adding a row $(\phi(\partial S/\partial x_j), \phi(\partial S/\partial a))$ where S is the relator $a^n l^s$. Since l is the boundary of a Seifert surface of K, $l \in G''$, hence $\phi(\partial l/\partial a) = \phi(\partial l/\partial x_j) = 0$. Therefore, $\phi(\partial S/\partial a) = \Lambda_n$ and $\phi(\partial S/\partial x_j) = 0$, hence the Alexander matrix of $\pi_1(M^3(K; n, s))$ is

$$\left[\begin{array}{c|c} \phi(\partial R_i/\partial x_j) & 0 \\ \hline 0 & \Lambda_n \end{array}\right]:$$

This completes the proof of Theorem 1.

THEOREM 2. *The first elementary ideal $E_1(M^3)$ of $\pi_1(M^3(K; n, s))$ is the principal ideal in $J(Z_n)$ generated by $\Delta_1(t)$, the first knot polynomial of K.*

PROOF OF THEOREM 2. The matrix obtained in Theorem 1 is an $(m + 1) \times (m + 1)$ matrix and $E_1(M^3)$ is generated by $m \times m$ subdeterminants of this matrix. Hence, $E_1(M^3)$ is generated by $\Delta_1(t)$ and $f_j(t)\Lambda_n$, $1 \leqslant j \leqslant \mu$, where $f_1(t), \ldots, f_\mu(t)$ are the nontrivial $(m - 1) \times (m - 1)$ subdeterminants of the Alexander matrix of K, that is, the generators of $E_2(K)$. Therefore,

$$E_1(M^3) = (\Delta_1(t)) + \Lambda_n \cdot E_2(K).$$

But

$$\Lambda_n \cdot E_2(K) = \Lambda_n \cdot \tau(E_2(K)) = \Lambda_n \cdot Z = (\Lambda_n).$$

Now, for any knot K, $\Delta_1(1) = \pm 1$, hence

$$\Lambda_n = (\Delta_1(1))^2 \Lambda_n = \Delta_1(t) \cdot \Delta_1(t) \cdot \Lambda_n \in (\Delta_1(t)).$$

Therefore,

$$E_1(M^3) = (\Delta_1(t)) + (\Lambda_n) = \Delta_1(t).$$

This completes the proof of Theorem 2.

We finish with two corollaries to Theorem 2. The first is an alternate proof of Theorem 1 of [**4**].

COROLLARY 1. *If K is a knot in S^3 with nontrivial polynomial $\Delta_1(t)$ then $M^3(K; n, s)$ is never topologically equivalent to $S^2 \times S^1$.*

PROOF. If $M^3(K; n, s) = S^2 \times S^1$ then $H_1(M^3) = \pi_1(M^3) = Z_0$; hence $n = 0$ and $s = \pm 1$. But the first elementary ideal of $J(Z_0)$ of the infinite cyclic group Z_0 is all of $J(Z_0)$ but, by Theorem 2, $E_1(M^3) = (\Delta_1(t))$ and $(\Delta_1(t)) \neq J(Z_0)$ since $\Delta_1(t)$ is nontrivial. Hence $\pi_1(M^3) \neq \pi_1(S^2 \times S^1)$, so $M^3(K; n, s) \neq S^2 \times S^1$.

Before stating the next corollary, we note that alternating knots with nontrivial polynomials satisfy the hypotheses. See [**1**] or [**5**].

COROLLARY 2. *Suppose K is a knot in S^3 with polynomial $\Delta_1(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_p t^p$. Let $\alpha = a_0 + a_2 + a_4 + \cdots$, that is, α is the sum of the coefficients of even powers of t in $\Delta_1(t)$. If $|\alpha| > 1$ then $\pi_1(M^3(K; n, s))$ is never a finite cyclic group of even order.*

PROOF. In this proof, $(\Delta_1(t))_n$ will denote the principal ideal in $J(Z_n)$ generated by $\Delta_1(t)$. Now if $\pi_1(M^3)$ is cyclic then $\pi_1(M^3) = H_1(M^3) = Z_n$; hence it suffices to show that $\pi_1(M^3(K; n, s)) \neq Z_n$ for even n. To show this it suffices to show that $(\Delta_1(t))_n \neq J(Z_n)$ for even n, since the first elementary ideal in $J(Z_n)$ of $\pi_1(M^3)$ is $(\Delta_1(t))_n$ and the first elementary ideal in $J(Z_n)$ of Z_n is all of $J(Z_n)$. But to show $(\Delta_1(t))_n \neq J(Z_n)$ for even n, it suffices to show $(\Delta_1(t))_2 \neq J(Z_2)$, because, for even n, there is a ring homomorphism of $J(Z_n)$ onto $J(Z_2)$ which takes $(\Delta_1(t))_n$ onto $(\Delta_1(t))_2$.

Now suppose the contrary, that is, suppose $(\Delta_1(t))_2 = J(Z_2)$. In $J(Z_2)$, $\Delta_1(t) = \alpha + (\varepsilon - \alpha)t$ where $\varepsilon = \pm 1$. Now if $(\Delta_1(t))_2 = J(Z_2)$, then there is an element $f(t)$ of $J(Z_2)$ such that $f(t)\Delta_1(t) = 1$. Say $f(t) = x + yt$ where x and y are integers. Then

$$1 = f(t)\Delta_1(t) = [\alpha x + (\varepsilon - \alpha)y] + [(\varepsilon - \alpha)x + \alpha y]t;$$

hence $\alpha x + (\varepsilon - \alpha)y = 1$ and $(\varepsilon - \alpha)x + \alpha y = 0$. Solving simultaneously, we obtain $x = \alpha/\varepsilon(2\alpha - \varepsilon)$ which cannot be an integer unless $|\alpha| \leqslant 1$, which contradicts the hypothesis.

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