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A SIMPLE NOETHERIAN RING NOT MORITA EQUIVALENT TO A DOMAIN

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ABSTRACT. An example of Zalesskii and Neroslavskii is used to produce an example of a simple ring that is not Morita equivalent to a domain.

In **[5]** an example is given of a simple Noetherian ring with divisors of zero but without nontrivial idempotents. In this note we show that this example also gives a negative answer to the important question of whether simple Noetherian rings are Morita equivalent to domains, thus answering **[1**, Question 1, p. 113].

The ring is constructed in the following way. Let k be a field of characteristic two. Define $R_1 = k(y)[X, X^{-1}]$ for indeterminates y and X. Let g be the $k(y)$-automorphism of R_1, defined by $g(X) = yX$ and let R_2 be the twisted group ring $R_1\langle g\rangle$; i.e. as an additive group, R_2 is isomorphic to the ordinary group ring but multiplication is defined by $rg = gr^g$ for $r \in R_1$. Let h be the $k(y)$-automorphism of R_2 defined by $h(X) = X^{-1}$ and $h(g) = g^{-1}$ and define S to be the twisted group ring $R_2\langle h\rangle$. The ring S was first constructed in **[5]**, where the following was proved.

THEOREM 1. *S is a simple Noetherian ring, not a domain, such that the only idempotents of S are* 0 *and* 1.

In **[5]**, S was actually defined as a localisation of a group ring over k. However, the characterisation given here is more convenient as it provides an easy method of calculating the Krull dimension of S, written Kdim S (for our purposes the following definition suffices. Given a prime ring R then Kdim $R = 1$ if R is not Artinian but R/I is an Artinian module for any essential one-sided ideal I).

THEOREM 2. Kdim $S = 1$.

PROOF. Clearly R_1 is hereditary. Since g leaves no ideal of R_1 invariant, R_2 is hereditary by **[3**, Theorem 2.3]. Thus Kdim $R_2 = 1$ (see for example **[2**, Theorem 1.3]). But S is finitely generated as a left or right R_2-module. So Kdim $S \leqslant 1$ and clearly we have equality.

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This result enables us to use the results of [**4**] to show that S has the properties described in the title.

THEOREM 3. *S is a simple Noetherian ring that is not Morita equivalent to a domain.*

PROOF. Suppose S is Morita equivalent to a domain A. Then $\text{Kdim } A = 1$ by Theorem 2. Let P be the image of S under the equivalence of the right module categories. Since S is not a domain, P is not isomorphic to a right ideal of A. Thus, with $\text{rk } P$ being the rank of the biggest free module that can be embedded in P, we have $\text{rk } P \geqslant 2 = 1 + \text{Kdim } A$. So by [**4**, Theorem 2.1], $P \cong Q \oplus A$ for some nonzero module Q. But then $S \cong I \oplus J$ for some nonzero right ideals I and J of S. This implies that S has nontrivial idempotents, which contradicts Theorem 1.

Since $(1 + h)S$ has a periodic projective resolution, S has infinite global dimension. Thus it is still possible that any simple Noetherian ring of finite global dimension is Morita equivalent to a domain. Indeed, using [**4**], it is possible to show that a simple Noetherian ring R, with finite global dimension and $\text{Kdim } R = 1$, is Morita equivalent to a domain.

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