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A NEW GENERALIZATION OF THE STURM COMPARISON THEOREM TO SELFADJOINT SYSTEMS

SHAIR AHMAD AND ALAN C. LAZER

ABSTRACT. The Sturm Comparison Theorem is generalized to second order linear systems. It is based on a comparison of the elements of the matrices involved.

In this paper, we consider the vector differential equations

$$y'' + P(t)y = 0 \tag{1}$$

and

$$z'' + Q(t)z = 0, \tag{2}$$

where $P(t) = (p_{ij}(t))$ and $Q(t) = (q_{ij}(t))$ are continuous $n \times n$ symmetric matrices on a given interval $[a, b]$. For the case $n = 1$ such equations have been studied extensively, beginning with the work of Sturm [7] in 1836. Since then there have been various extensions of the Sturmian theory to selfadjoint systems of second order linear differential equations, initiated by Morse [5] in 1930. Further extensions were subsequently given by Birkhoff and Hestenes, Reid, and others (see [6]). It was shown in [5] that if $P(t)$ and $Q(t)$ are symmetric, $Q(t) \geqslant P(t)$, i.e. $Q(t) - P(t)$ is positive semidefinite, with $Q(\bar{t}) > P(\bar{t})$ for some number $\bar{t}$ in the interval $[a, b]$, and if (1) has a nontrivial solution $y(t)$ satisfying $y(a) = y(b) = 0$, then (2) has a nontrivial solution $z(t)$ such that $z(a) = z(c) = 0$, where c is some number in the open interval (a, b). The purpose of this note is to present a similar theorem which is based on an elementwise comparison of the matrices $P(t)$ and $Q(t)$. Our theorem neither implies the theorem of Morse nor is it implied by it. We are able to give a relatively simple proof based on variational methods and an earlier result of ours.

We recall that a number b, $b > a$, is said to be *conjugate* to a relative to a certain equation of the form (1) if there exists a nontrivial solution of this equation which vanishes at a and b. The equation is said to be *disconjugate* on an interval I if no nontrivial solution of it vanishes more than once in I. It is well-known that in the selfadjoint case, i.e. if $P(t)$ is symmetric, conjugate points are isolated. Let $\mathcal{Q}[a, b]$ denote the set of absolutely continuous R^n-valued functions $h(t)$ on $[a, b]$ such that $|h'| \in L^2[a, b]$ and $h(a) = h(b)$

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= 0. The proof of the following lemma, which essentially follows from Reid [**6**, p. 332], has been given in [**2**]. For more information on the preceding concepts one might consult [**4**] and [**6**].

LEMMA 1. *Let $A(t)$ be a continuous $n \times n$ symmetric matrix on $[a, b]$, and let $J[h; a, b]$ denote the functional*

$$J[h; a, b] = \int_a^b (\langle h', h'\rangle - \langle Ah, h\rangle)\, dt$$

over the set $\mathcal{Q}[a, b]$ of admissible functions. Then $J[h; a, b] \geqslant 0$ for all h in $\mathcal{Q}[a, b]$ if and only if the interval $[a, b]$ contains no point conjugate to a in its interior relative to the equation $x'' + A(t)x = 0$.

REMARK 1. The above lemma is a slight modification of the well-known fact (see, e.g., [**3**] or [**6**]) that the equation

$$x'' + A(t)x = 0 \tag{3}$$

is disconjugate on the interval $[a, b]$ if and only if

$$J[h; a, b] = \int_a^b (\langle h', h'\rangle - \langle Ah, h\rangle)\, dt > 0$$

over the set of admissible functions h, $h \not\equiv 0$.

THEOREM 1. *Assume that in equations* (1) *and* (2), $q_{ij}(t) \geqslant p_{ij}(t)$ *for* $1 \leqslant i, j \leqslant n$, *and* $t \in [a, b]$. *Further, assume that* $q_{ii}(\bar{t}) > p_{ii}(\bar{t})$ *for some* $\bar{t} \in (a, b)$, $1 \leqslant i \leqslant n$, *and that* $p_{ij}(t) \geqslant 0$ *for* $i \neq j$. *If* (1) *has a nontrivial solution* $y(t)$ *satisfying* $y(a) = y(b) = 0$, *then* (2) *has a nontrivial solution* $z(t)$ *such that* $z(a) = z(c) = 0$, $a < c < b$.

We note that in the above theorem only the diagonal elements of Q are required to be strictly greater than those of P at a point $\bar{t}$. It can be verified, by letting $P(t) = \text{diag}(0, 1)$ and $Q(t) = \text{diag}(1, 1)$, that this condition cannot be relaxed any further.

PROOF OF THEOREM 1. First we assume that b is the first conjugate point of a relative to (1). By Theorem 1 of [**1**] (or Theorem 2 of [**2**]) it follows that there exists a nontrivial solution

$$u(t) = \text{col}(u_1, \ldots, u_n)$$

of (1) such that $u(a) = u(b) = 0$ and $u_j(t) \geqslant 0$ on $[a, b]$, $j = 1, \ldots, n$. Let $J[h; a, b]$ and $\tilde{J}[h; a, b]$ define the functionals given by

$$J[h; a, b] = \int_a^b (\langle h', h'\rangle - \langle Ph, h\rangle)\, dt$$

and

$$\tilde{J}[h; a, b] = \int_a^b (\langle h', h'\rangle - \langle Qh, h\rangle)\, dt$$

over the set $\mathcal{Q}[a, b]$ of admissible functions. Then,

$$\begin{aligned}\tilde{J}[u; a, b] &= \int_a^b (\langle u', u'\rangle - \langle Qu, u\rangle)\, dt \\ &= \int_a^b \left(\langle u', u'\rangle - \sum_{i=1}^n q_{ii}(t)u_i^2 - \sum\sum_{i\neq j} q_{ij}(t)u_iu_j \right) dt \\ &< \int_a^b \left(\langle u', u'\rangle - \sum_{i=1}^n p_{ii}(t)u_i^2 - \sum\sum_{i\neq j} p_{ij}u_iu_j \right) dt \\ &= \int_a^b (\langle u', u'\rangle - \langle P(t)u, u\rangle)\, dt = J[u; a, b].\end{aligned}$$

The above inequality follows from the hypothesis that $p_{ii}(\bar{t}) < q_{ii}(\bar{t})$, $1 \leqslant i \leqslant n$. Multiplying the equation

$$u'' + P(t)u = 0$$

by $-u$ and integrating from a to b we see that

$$\int_a^b (\langle u', u'\rangle - \langle Pu, u\rangle)\, dt = 0.$$

We have thus shown that $\tilde{J}[u; a, b] < J[u; a, b] = 0$. By Lemma 1 a has a conjugate point c in the open interval (a, b) relative to equation (2). Consequently, there exists a nontrivial solution $z(t)$ of (2) satisfying $z(a) = z(c) = 0$, where c is some number in the open interval (a, b).

Now, let us assume that b is not the first conjugate point of a relative to (1). Let $\eta(a)$ be the first conjugate point of a, and let

$$v(t) = \mathrm{col}(v_1, \ldots, v_n)$$

be a solution of (1) such that $v(a) = v(\eta(a)) = 0$ and $v_j(t) \geqslant 0, j = 1, \ldots, n$. Then $a < \eta(a) < b$. The same argument that we gave to establish that $\tilde{J}[u; a, b] < J[u, a, b]$ shows that

$$\tilde{J}[v; a, \eta(a)] \leqslant J[v; a, \eta(a)].$$

We no longer have strict inequality since we can not assume that $\bar{t} \in [a, \eta(a)]$. However, $\tilde{J}[v; a, \eta(a)] \leqslant J[v; a, \eta(a)] = 0$ implies (see Remark 1) that (2) is not disconjugate on $[a, \eta(a)]$. Hence the interval $[a, \eta(a)]$ contains a point conjugate to a relative to (2). For, it follows (see [3] or [6]) that if $\tilde{\eta}(a)$ is the first conjugate point of a relative to (2) then (2) is disconjugate on $[a, \tilde{\eta}(a))$. Hence we must have $\tilde{\eta}(a) \leqslant \eta(a)$. This shows that (2) has a nontrivial solution $z(t)$ such that $z(a) = z(c) = 0$, where c is some number in the interval $(a, \eta(a)]$, and the proof is complete.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74074

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI, CINCINNATI, OHIO 45221