PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 68, Number 2, February 1978

EXTREMAL HOLOMORPHIC IMBEDDINGS BETWEEN THE BALL AND POLYDISC[1]

H. ALEXANDER

ABSTRACT. The following problem of Fornaess and Stout is considered: Find, among all polydiscs imbedded in the unit ball, the one which contains the largest ball centered at the origin.

Fornaess and Stout [1] have observed (as a consequence of their result that a monotone union of (imbedded) polydiscs in a taut complex manifold is (biholomorphic to) a polydisc) that an imbedded polydisc in the unit ball in $\mathbf{C}^m$ ($m > 1$) cannot contain balls centered at the origin of radius arbitrarily close to one. More precisely, there is a positive number $R_0(m) < 1$ such that the image of the unit polydisc U^m by an imbedding in the unit ball B_m in $\mathbf{C}^m$ never contains sB_m for $s > R_0(m)$, where, for $W \subseteq \mathbf{C}^m$, $sW = \{sz: z \in W\}$. Nevertheless, B_m contains an imbedded polydisc of full measure [1]. Fornaess and Stout, by exhibiting the map $F_0: U^m \to B_m$, $F_0(z) = z/\sqrt{m}$ with $F_0(U^m) \supseteq B_m/\sqrt{m}$, showed that $R_0(m) \geqslant 1/\sqrt{m}$ and raised the problem of finding the smallest (or any specific) value of $R_0(m)$. We show that, in fact, the smallest value of $R_0(m)$ is $1/\sqrt{m}$ and that F_0 is the unique extremal imbedding, up to automorphisms.

PROPOSITION 1. *If F: $U^m \to B_m$ is a holomorphic imbedding for which $F(U^m) \supseteq sB_m$, then $s \leqslant 1/\sqrt{m}$. Moreover, the equality $s = 1/\sqrt{m}$ holds if and only if $F = B \circ F_0 \circ A$ where A is a biholomorphism of U^m and B is a unitary transformation.*

COROLLARY 1. *The polydisc $U^m/\sqrt{m} \subseteq B_m$ is maximal among imbedded polydiscs in B_m.*

The reasoning of Fornaess and Stout also shows that there is a positive number $S_0(m) < 1$ such that if B_m is imbedded into U^m, then the image does not contain sU^m for $s > S_0(m)$. Now consider the inclusion map G_0: $B_m \to U^m$, $G_0(z) = z$. Then $G_0(B_m) \supseteq U^m/\sqrt{m}$ and so $S_0(m) \geqslant 1/\sqrt{m}$. This is the extremal case:

PROPOSITION 2. *If G: $B_m \to U^m$ is a holomorphic imbedding such that $G(B_m) \supseteq sU^m$, then $s \leqslant 1/\sqrt{m}$. Moreover the equality $s = 1/\sqrt{m}$ holds if and only if $G = G_0 \circ B (\equiv B)$ where B is a biholomorphism of B_m.*

Received by the editors June 20, 1977.
AMS (MOS) subject classifications (1970). Primary 32H99.
[1]Research supported in part by NSF 76 07287 A01.

COROLLARY 2. *The ball $B_m \subseteq U^m$ is maximal among imbedded balls in the polydisc U^m.*

PROOF OF PROPOSITION 1. Let $\|z\|$ be the Euclidean norm for $z \in \mathbf{C}^m$. Assume, first, that $F(0) = 0$. Write F in a vector Taylor series: $F(z) = \Sigma a_\alpha z^\alpha$, where $a_\alpha \in \mathbf{C}^m$. The almost everywhere defined boundary values of the bounded holomorphic function F will also be denoted by F. Since $\liminf \|F(re^{i\theta}, 0, \ldots, 0)\| \geqslant s$ as $r\uparrow 1$, we have

$$s^2 \leqslant \frac{1}{2\pi} \int_0^{2\pi} \|F(e^{i\theta}, 0, 0)\|^2 \, d\theta = \sum \left\{ \|a_\alpha\|^2 : \alpha \in \mathbb{S}_1 \right\}$$

where $\mathbb{S}_k = \{\alpha = (\alpha_1, \ldots, \alpha_m)\colon \ \alpha_k > 0, \ \alpha_j = 0 \ \text{for} \ j \neq k\}$ for $k = 1, 2, \ldots, m$. In the same way we get $s^2 \leqslant \Sigma\{\|a_\alpha\|^2 \colon \alpha \in \mathbb{S}_k\}$. Adding these m inequalities, we have

$$\begin{aligned} ms^2 &\leqslant \sum \left\{ \|a_\alpha\|^2 : \alpha \in \mathbb{S}_1 \cup \mathbb{S}_2 \cup \cdots \mathbb{S}_m \right\} \\ &\leqslant \sum \|a_\alpha\|^2 \leqslant \int_{T^m} \|F\|^2 \, dh \leqslant 1, \end{aligned}$$

where h is Haar measure on the torus T^m; we are using $a_\alpha = 0$ for $\alpha = 0$ and $\|F\|^2 \leqslant 1$ a.e. on T^m. Thus $s \leqslant 1/\sqrt{m}$.

In the case of equality $s = 1/\sqrt{m}$ we have (i) $a_\alpha = 0$ for $\alpha \notin \mathbb{S}_1 \cup \mathbb{S}_2 \cup \cdots \cup \mathbb{S}_m$, (ii) $\|F(0, \ldots, e^{i\theta}, \ldots, 0)\| = 1/\sqrt{m}$ a.e. on the unit circle, where $e^{i\theta}$ is in the kth position, for $1 \leqslant k \leqslant m$, and (iii) $\|F\| = 1$ a.e. on T^m. By (i) and (ii) we can write $F(z_1, z_2, \ldots, z_m) = F_1(z_1) + F_2(z_2) + \cdots + F_m(z_m)$ for $F_k\colon U \to \mathbf{C}^m$ with $F_k(0) = 0$ and $\|F_k(e^{i\theta})\| = 1/\sqrt{m}$ a.e. on the circle; abusing notation, we shall also view F_k as a mapping defined on U^m which depends only on the kth variable.

For $z, w \in \mathbf{C}^m$ put $\langle z, w\rangle = \Sigma z_j\overline{w}_j$, the standard Hermitian inner product. Then $\operatorname{Re}\langle z, w\rangle$ is the standard real inner product on $R^{2m} = \mathbf{C}^m$. Now (ii) and (iii) imply for almost all $p = (e^{i\theta_1}, \ldots, e^{i\theta_m}) \in T^m$,

$$\begin{aligned} 1 &= \|F(p)\|^2 = \sum \left\langle F_k(e^{i\theta_k}), F_j(e^{i\theta_j}) \right\rangle \\ &= \sum_1^m \|F_k(p)\|^2 + 2 \operatorname{Re} \sum_{k<j} \left\langle F_k(e^{i\theta_k}), F_j(e^{i\theta_j}) \right\rangle \\ &= \sum_1^m \frac{1}{m} + 2\, Q(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_m}) \end{aligned}$$

where $Q(z_1, \ldots, z_m) = \operatorname{Re} \Sigma_{k<j}\langle F_k(z_k), F_j(z_j)\rangle$. Thus $Q = 0$ a.e. on T^m. As Q is a bounded m-harmonic function on U^m, it follows (see [**2**]) that $Q \equiv 0$ on U^m. Thus $\operatorname{Re}\langle F_k(z_k), F_j(z_j)\rangle = Q(0, \ldots, z_k, \ldots, z_j, \ldots, 0) = 0$ for $|z_j| < 1$, $|z_k| < 1$; i.e., $F_k(z_k)$ is real orthogonal to $F_j(z_j)$. Define H_j to be the *real* linear span in $\mathbf{C}^m$ of the set $F_j(U)$. We have proved that the H_j are mutually (real) orthogonal. Since F is one-to-one, it follows that each F_j is one-to-one on U. Therefore, $\dim_{\mathbf{R}} H_j \geqslant 2$. We conclude that $\dim_{\mathbf{R}} H_j = 2$ for each j and that H_j is complex linear (a "complex line"). Now, replacing F by $V \circ F$

where V is a unitary transformation taking H_j to the jth complex coordinate axis, we may assume that H_j *is* the jth coordinate axis. Thus $F_j(z_j) = (0, \ldots, f_j(z_j), \ldots, 0)$ where, in the jth position, there is a bounded complex valued holomorphic function f_j defined on U and satisfying $f_j(0) = 0$ and $|f_j(e^{i\theta})| = 1/\sqrt{m}$ a.e.. Thus $\sqrt{m}\ f_j$ is an inner function. Since f_j is one-to-one we conclude that $f_j(\zeta) = e^{i\beta_j}\zeta/\sqrt{m}$ for real β_j. This proves that $F = B \circ F_0$ where B is a unitary transformation.

We have assumed that $F(0) = 0$. The general case can be reduced to this case by preceding F by an automorphism of U^m.

Proof of Corollary 1. If P is an imbedded polydisc in B_m containing $U^m/\sqrt{m}$, then P contains $B_m/\sqrt{m}$ and by Proposition 1, there is a unitary transformation B such that $P = B(U^m/\sqrt{m})$. It follows that $B(T^m/\sqrt{m}) = T^m/\sqrt{m}$ and hence $|z_k| \leqslant 1/\sqrt{m}$ on P. Therefore $P = U^m/\sqrt{m}$.

Proof of Proposition 2. By preceding G by an automorphism of B_m, we may assume, without loss of generality, that $G(0) = 0$. Let $\varepsilon > 0$. Let $F = (G^{-1}|sU^m)$: $sU^m \to B_m$. Applying Proposition 1 to F, we see that $F(sU^m)$ does not contain $(1 + \varepsilon)/\sqrt{m}\ B_m$. Thus there is a point $p \in B_m \setminus F(sU^m)$ with $\|p\| < (1 + \varepsilon)/\sqrt{m}$. Hence $q \equiv G(p) \notin sU^m$ and so one of the coordinates, say q_k, of q satisfies $|q_k| \geqslant s$. Schwarz's lemma applied to the kth component g_k of G yields $|g_k(z)| \leqslant \|z\|$ for all $z \in B_m$. Thus $s \leqslant |q_k| = |g_k(p)| \leqslant \|p\| < (1 + \varepsilon)/\sqrt{m}$; i.e., $s \leqslant 1/\sqrt{m}$.

In order that $s = 1/\sqrt{m}$, $F(U^m/\sqrt{m})$ must contain $B_m/\sqrt{m}$. From Proposition 1 we conclude that F is a unitary transformation (restricted to $U^m/\sqrt{m}$). This implies that G has the desired form.

Corollary 2 follows directly from Proposition 2.

References

1. J. E. Fornaess and E. L. Stout, *Polydiscs in complex manifolds*, Math. Ann. **227** (1977), 145–153.

2. Walter Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.

Department of Mathematics, University of Illinois at Chicago Circle, Chicago, Illinois 60680