PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 68, Number 2, February 1978

THE VOLUME OF A SLIGHTLY CURVED SUBMANIFOLD IN A CONVEX REGION

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ABSTRACT. Let T be a compact convex region in an n-dimensional Riemannian space, k_s be the minimum sectional curvature in T, and $\kappa > 0$ be the minimum normal curvature of the boundary of T. Denote by $P^\nu(\xi)$ a ν-dimensional sphere, plane or hyperbolic plane of curvature ξ. We assume that k_s, κ are such that on $P^2(k_s)$ there exists a circumference of curvature κ. Let $R_0 = R_0(\kappa, k_s)$ be its radius. Now, let Q be a convex (in interior sense) m-dimensional surface in T whose normal curvatures with respect to any normal are not greater than χ satisfying $0 \leqslant \chi < \kappa$. Denote by L_χ the length of a circular arc of curvature χ in $P^2(k_s)$ with the distance $2R_0$ between its ends. We prove that the volume of Q does not exceed the volume of a ball in $P^m(k_s - (n-m)\chi^2)$ of radius $\frac{1}{2}L_\chi$. These volumes are equal when T is a ball in $P^n(k_s)$ and Q is its m-dimensional diameter.

1. Statement of the result. Let M be an n-dimensional Riemannian space, $n \geqslant 2$, of regularity class C^4. Throughout this paper, it will be assumed that manifolds, their edges, surfaces, mappings etc. all are of class C^4 without a special mention. We consider in M a connected region that has a compact closure T and is bounded by a nonempty hypersurface Γ. We suppose that all the normal curvatures of Γ on the side of the interior normal are not less than some positive number κ. (Hence, Γ is two-sided and T is situated on one side of Γ.) Suppose that in the compact region T the sectional curvatures $\geqslant k_s > -\kappa^2$. In such a case we will call T *normal.*

Denote by $P^m(k_s)$ an m-dimensional sphere, Euclidean or hyperbolic space of curvature k_s. It was remarked in [3, §1], that on $P^2(k_s)$ there exists a circle T_0 whose circumference Γ_0 has geodesic curvature κ if and only if $k_s > -\kappa^2$.

Denote by Q a connected m-dimensional Riemannian manifold, $2 \leqslant m < n$, with an edge (possibly empty) $\partial Q \subset Q$. We call Q *convex* if for any p, $q \in Q, p \neq q$, there exists a minimal geodesic with the ends p, q.

An isometric immersion i: $Q \to T$ will be called χ-*curved*, $\chi \geqslant 0$, if the curvature of the image of any geodesic in Q under the mapping i does not exceed χ.

Let $\chi < \kappa$ and i: $Q \to T$ be a χ-curved immersion. It follows immediately from the corollary of Theorem 1 in [3, §1], that any geodesic in Q is not longer than the length L_χ of a circular arc in the circle T_0 which has geodesic

Received by the editors August 4, 1976.
AMS (MOS) subject classifications (1970). Primary 53C40.
[1]Supported by NRC of Canada Grants awarded to Professors H. S. M. Coxeter, W. H. Greub, P. Scherk, D. K. Sen and J. R. Vanstone.

curvature χ and whose ends are opposite points of Γ_0. Therefore, in fact,

$$\partial Q \neq \varnothing. \tag{1.1}$$

If Q is convex, then obviously the diameter D of Q satisfies

$$D \leqslant L_\chi. \tag{1.2}$$

We prove the following

THEOREM. *Let a manifold Q be convex, $\chi < \kappa$ and $i\colon Q \to T$ be a χ-curved immersion. Then* (i) *Q is a compact manifold with a nonempty edge and* (ii) *volume of Q does not exceed the volume of a ball B of radius $\frac{1}{2} L_\chi$ in $P^m(k_s - (n-m)\chi^2)$.*

REMARKS. 1. The statement (i) has already been proved in the preceding remarks (see (1.1) and (1.2)).

2. Q and B have equal volumes when T is a ball in $M = P^n(k_s)$ and Q is an m-dimensional diameter of T. (Then $\chi = 0$.)

3. If $P^m(k_s - (n-m)\chi^2)$ for $m = 1$ is regarded as $R^1 = R$, then this theorem generalizes the corollary in [**3**, §1].

4. If Q is not convex, then its volume can be arbitrarily large.

5. If $\kappa \leqslant 0$ or $k_s \leqslant -\kappa^2$, then, in such a region, the submanifold Q can have arbitrarily large diameter and volume, even if the immersion i is 0-curved.

6. The smoothness requirement (C^4) is not really essential. It only allows for the use of results in [**1**], [**2**], [**3**] and could be reduced to C^2 by means of a suitable approximation.

7. It is easy to see that B is not more than a semisphere of $P^m(k_s - (n-m)\chi^2)$ when $k_s - (n-m)\chi^2 > 0$.

8. If $\frac{1}{2} L_\chi$ in this Theorem is replaced by L_χ, then such a rougher estimate can be obtained in a simpler way: instead of the remote point q defined in §2, we can use then an arbitrary point of Q.

2. Proof of Theorem. In view of Remark 1, we need to prove only (ii).

LEMMA 1. *Sectional curvatures of the manifold $Q \setminus \partial Q$ are not less than $k_s - (n-m)\chi^2$.*

PROOF. Let $q \in Q \setminus \partial Q$ and σ be a 2-dimensional direction at the point q. Denote by σ_* the image of σ under the mapping $i_{*q}\colon Q_q \to M_{i(q)}$. Let $k(\sigma)$ and $k(\sigma_*)$ be the curvatures of the manifolds Q and M in the directions σ and σ_*. From formula (9) in [**4**, point 3.7], we have that:

$$k(\sigma) = k(\sigma_*) + \sum_{\mu=1}^{n-m} \begin{vmatrix} l_\mu(u,u) & l_\mu(u,v) \\ l_\mu(u,v) & l_\mu(v,v) \end{vmatrix} \tag{2.1}$$

where $u, v \in \sigma$ are orthogonal unit vectors and $l_\mu(\cdot\,,\cdot)$ is the second quadratic form of Q with respect to a unit normal N_μ, $\mu = 1, 2, \ldots, n-m$, satisfying $N_\mu \perp N_\nu$ when $\mu \neq \nu$.

Denote by u_μ, v_μ orthogonal unit vectors showing the principal directions of the form l_μ restricted to σ. Then

$$(2.2)\qquad \begin{vmatrix} l_\mu(u,u) & l_\mu(u,v) \\ l_\mu(u,v) & l_\mu(v,v) \end{vmatrix} = \begin{vmatrix} l_\mu(u_\mu,u_\mu) & 0 \\ 0 & l_\mu(v_\mu,v_\mu) \end{vmatrix} \geqslant -\chi^2.$$

It follows from the inequalities $k(\sigma_*) \geqslant k_s$ and (2.2) that $k(\sigma) \geqslant k_s - (n-m)\chi^2$. Q.E.D.

We will denote by $\rho(\cdot,\cdot)$ the intrinsic distance between the sets in the region T, and by ab we will denote a shortest path with the ends $a, b \in T$. We will also use the notation ab for $\rho(a,b)$. A shortest path ab always exists and it is a geodesic. This fact follows from an argument analogous to that contained in Lemma 2 and the remark after it in **[1]**.

By compactness of Q, there exists a point $q \in Q$ such that $\rho(i(q),\Gamma) = \max_{p\in Q}\rho(i(p),\Gamma)$. We will call q the *remote* point of Q. Obviously, $i(q) \in \operatorname{int} T$ because otherwise $i(Q) \subset \Gamma$ and the image under the immersion i of any geodesic in Q has the curvature not less than κ. This is obviously impossible by the definition of χ-curved immersion i, and by the fact that $\chi < \kappa$.

Let $b \in T$. A shortest path ba, $a \in \Gamma$, will be called *b-projecting* if $ba = \rho(b,\Gamma)$.

LEMMA 2. *Let q be a remote point of Q and let $b = i(q)$. Denote by $\Omega \subset Q_q$ the set of directions (unit vectors) of the geodesic emanating from q toward the interior of Q.*[2]

Then for any $u \in \Omega$ there exists a b-projecting shortest path which forms an angle $\phi \leqslant \pi/2$ with the vector $i_*(u)$.

The proof of Lemma 2 will be given in §3. Meanwhile we continue with the proof of the theorem.

Obviously Ω is homeomorphic to an open $(m-1)$-dimensional circle when $q \in \partial Q$. Denote by C the cut locus of Q from the point q. Let $u \in \Omega$ and $z(u) > 0$ be the minimum number such that $\exp_q(z(u)\cdot u) \in C \cup \partial Q$. $z(u)$ exists since the set $C \cup \partial Q$ is closed (because $Q \setminus (C \cup \partial Q)$ is obviously open).

We want to show first that

$$(2.3)\qquad z(u) \leqslant \tfrac{1}{2} L_\chi, \qquad u \in \Omega.$$

By Lemma 2, there exists a b-projecting shortest path ba which forms an angle $\phi \leqslant \pi/2$ with the curve $g\colon [0, z(u)] \to T$ given by the formula

$$(2.4)\qquad g(l) = i(\exp_q(lu)).$$

Applying the inequality (1.4) in **[3]** (where $\overline{L}_0(\pi/2)$ is obviously $\frac{1}{2} L_\chi$) to the curve g of curvature $\leqslant \chi$ and the mentioned path ba we obtain (2.3).

Let us put $\Phi = \bigcup lu$, where $u \in \Omega$ and $0 < l < z(u)$. It is easy to see that

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[2]We mean that all points of the geodesic sufficiently close to q and different from q lie in int T. If $q \notin \partial Q$, then $\Omega = S^m$.

for any $p \in \operatorname{int} Q, p \neq q$, the minimum geodesic qp emanates from q toward the interior of Q. Therefore

$$\Psi \stackrel{\text{def}}{=} \exp_q(\Phi) = Q \setminus (q \cup \partial Q \cup C)$$

and the mapping $\exp_q$ restricted to Φ is one-to-one. By an argument contained in Lemma 8 in [2] we can see that the set C has m-dimensional measure zero. Therefore

$$V(Q) = V(\Psi), \tag{2.5}$$

where $V(\cdot)$ means volume.

Let $\tilde{q}$ be the center of the ball B in the space $\tilde{Q} \stackrel{\text{def}}{=} P^m(k_s - (n-m)\chi^2)$ and $j\colon Q_q \to \tilde{Q}_{\tilde{q}}$ be an isometry. Let us put

$$f = \exp_{\tilde{q}} \circ j \circ \exp_q^{-1}\colon \Psi \to \tilde{Q}. \tag{2.6}$$

By (2.3), the set $\tilde{\Psi} \stackrel{\text{def}}{=} f(\Psi) \subset B$ so that

$$V(\tilde{\Psi}) \leqslant V(B). \tag{2.7}$$

On the strength of (2.5) and (2.7), the proof will be completed if we show that

$$V(\Psi) \leqslant V(\tilde{\Psi}). \tag{2.8}$$

To show this it is enough to establish that the induced mapping $f_*\colon T\Psi \to T\tilde{\Psi}$ does not decrease the length of tangent vectors. Really, if it is proved then the mapping f does not decrease volumes since, for any point $p \in \Psi$, the mapping $f_{*p}\colon \Psi_p \to \tilde{\Psi}_{\tilde{p}}$ ($\tilde{p} \stackrel{\text{def}}{=} f(p)$), like any linear transformation, can be reduced to expansions into m pair-wise orthogonal directions and to orthogonal transformations.

Let $p \in \Psi$ and a vector $v \in \Psi_p$. We need to show that $|f_*(v)| \geqslant |v|$. Let the vectors $v^=$ and $v^\perp$ be such that $v = v^= + v^\perp$, $v^= \perp v^\perp$, and $v^=$ is directed along the shortest path pq. It follows easily from the definition of f that $f_*(v^=)$ is directed along $\tilde{p}\tilde{q}$, $|f_*(v^=)| = |v^=|$, and $f_*(v^\perp) \perp f_*(v^=)$. Since $f_*(v) = f_*(v^=) + f_*(v^\perp)$, it will be enough to prove that $|f_*(v^\perp)| \geqslant |v^\perp|$.

Let $p(t)$, $t \in [0, \varepsilon]$, be a curve such that $p(0) = p$, $\dot{p}(0) = v^\perp$ and $qp(t) = qp$. We put $\tilde{p}(t) = f(p(t))$. Obviously, $\tilde{q}\tilde{p}(t) = qp(t) = qp$, and $\measuredangle\, \tilde{p}\tilde{q}\tilde{p}(t) = \measuredangle\, pqp(t)$. By Rauch comparison theorem we have: $|\dot{\tilde{p}}(0)| \geqslant |\dot{p}(0)|$, i.e. $|f_*(v^\perp)| \geqslant |v^\perp|$. This completes the proof of the theorem.

3. Proof of Lemma 2. Suppose otherwise. That is suppose that some $u \in \Omega$ forms an angle $> \pi/2$ with any b-projecting shortest path. We consider the curve g given by (2.4). Let $l_i \xrightarrow[i\to\infty]{} 0$, $l_i > 0$, and $b_i = g(l_i)$, $i = 1, 2, \ldots$. Denote by $b_i a_i$ a b_i-projecting shortest path. By a simple compactness argument (selection of a subsequence), we may regard the sequence $b_i a_i$ as converging to a b-projecting shortest path ba.

We first want to prove the following statement

(α) there are no conjugate points on the segment ba.

Let us extend ba beyond its end a (as a geodesic) to a close point d such that the extension ad is still a shortest path and $S \cap T = a$ where S is the sphere of radius ad with the center at d (see the figure). If (α) is not true, then there exists an arc $\overset{\smile}{bd}$ satisfying

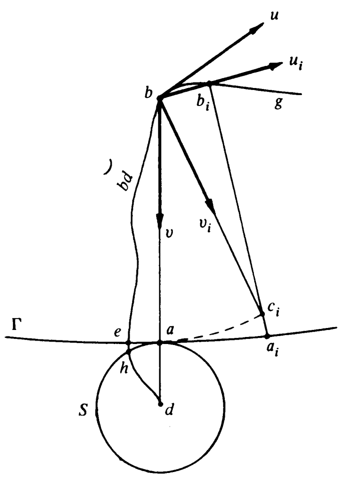
$$l(\overset{\smile}{bd}) < ba + ad \tag{3.1}$$

where $l(\cdot)$ means the length. In what follows we will denote by $\overset{\smile}{\cdot\cdot}$ parts of the arc $\overset{\smile}{bd}$. Let $e \in \overset{\smile}{bd}$, $\overset{\smile}{be} \subset T$, $e \in \Gamma$ and let $h \in \overset{\smile}{bd} \cap S$. We have:

$$l(\overset{\smile}{be}) \leqslant l(\overset{\smile}{bh}), \tag{3.2}$$

$$l(\overset{\smile}{hd}) \geqslant ad. \tag{3.3}$$

Subtracting (3.3) from (3.1) we obtain that $l(\overset{\smile}{bh}) < ba$. Combining it with (3.2) we find that $l(\overset{\smile}{be}) < ba$ which is impossible by definition of b-projecting shortest path ba. So, (α) has been proved.



For sufficiently large i the inequality $bb_i < ba \leqslant ba_i$ holds. Therefore on $b_i a_i$ there exists a point c_i such that $bc_i = ba$. Let us connect b and c_i with a shortest path bc_i and consider the triangle bb_ic_i (see the figure). The direction $u_i \in \Omega$ of its side bb_i converges to u. The relation $a_i \to a$ obviously implies $c_i \to a$. This fact together with (α) means that the direction v_i of the side bc_i converges to the direction v of ba.

Let us regard the directions u_i, v_i and the length bb_i as three independent variables. As soon as $bc_i = ba$ does not depend on i and because of (α), there exists $\varepsilon > 0$ such that the length b_ic_i is a regular function $\lambda(u_i, v_i, bb_i)$ in the compact region σ: $|u_i - u| \leqslant \varepsilon$, $|v_i - v| \leqslant \varepsilon$, $|bb_i| \leqslant \varepsilon$. (Negative bb_i means the displacement in the direction $-u_i$.) By Hadamard Lemma,

$$\lambda(u_i, v_i, bb_i) = \lambda(u_i, v_i, 0) + \frac{\partial \lambda}{\partial bb_i}(u_i, v_i, 0) \cdot bb_i + \theta(u_i, v_i, bb_i) \cdot (bb_i)^2 \tag{3.4}$$

where the function θ is regular in σ and therefore bounded. Let C be such that $|\theta| < C$. For sufficiently small ε, the segment bc_i being close to ba does not contain conjugate points. Therefore $\partial\lambda(u_i, v_i, 0)/\partial bb_i = -\langle u_i, v_i\rangle$. Obviously, $\lambda(u_i, v_i, 0) = ba$.

Now, for large i, the point $(u_i, v_i, bb_i) \in \sigma$ and we get from (3.4) that

$$b_i c_i > ba + bb_i(-\langle u_i, v_i\rangle - C \cdot bb_i). \tag{3.5}$$

By our original contrary assumption, $\langle u, v\rangle < 0$. Therefore $\lim_{i\to\infty}\langle u_i, v_i\rangle = \langle u, v\rangle < 0$. Since $bb_i \underset{i\to\infty}{\to} 0$, we find from (3.5) that $b_i c_i > ba$ for sufficiently large i. Consequently, $b_i a_i > ba$. But then the point q ($q \in i^{-1}(b)$) is not remote. This contradiction proves the Lemma.

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