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ON THE SEMI-CANONICAL PROPERTY IN THE PRODUCT SPACE $X \times I$

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ABSTRACT. As one of the several properties in generalized metric spaces, the semi-canonical property has been discussed from the viewpoint of the extension of mappings. In this paper, that property will be discussed in product space $X \times I$ and reduced to a property of X.

1. Introduction. By a pair (X, A) we mean a topological space X with a closed subset A of X. Let (X, A) be a pair. As in [6], a collection $\mathcal{V} = \{V_\lambda: \lambda \in \Lambda\}$ of open subsets of X is called a *semi-canonical cover for* (X, A) if

(1) $\bigcup_{\lambda \in \Lambda} V_\lambda = X - A$, and

(2) for each $x \in A$ and each neighborhood U of x in X there exists a neighborhood W of x in X such that $\mathrm{St}(W, \mathcal{V}) \subset U$, where

$$\mathrm{St}(W, \mathcal{V}) = \bigcup \{V \in \mathcal{V}: V \cap W \neq \varnothing\}$$

denotes the star of W with respect to $\mathcal{V}$.

If a semi-canonical cover exists for (X, A), (X, A) is called a *semi-canonical pair.*

It was proved by D. Hyman ([6], [7]) that (X, A) is a semi-canonical pair if X is the image of a metric space by a closed continuous map. It is also mentioned by M. Cauty [3] that, if X is a stratifiable space (cf. [2]), then any pair (X, A) is semi-canonical. However, quite recently S. San-ou [11] pointed out that Cauty's statement was false by constructing an M_1-space X (cf. [4]) such that (X, A) was not semi-canonical for some closed subset A of X.

The purpose of this paper is to discuss the semi-canonical property in the product space $X \times I$ of a T_1 space X with the unit closed interval I and to reduce it to a property in X.

THEOREM 1. *Let X be a T_1 space. Then $(X \times I, X \times \{0\})$ is a semi-canonical pair if and only if X is metrizable.*

By Theorem 1 it can be easily seen that, if X is any nonmetrizable M_1-space, then $X \times I$ is an M_1-space such that $(X \times I, X \times \{0\})$ is never semi-canonical.

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THEOREM 2. *Let X be a T_1 space. Then $(X \times I, K \times \{0\})$ is a semi-canonical pair for each compact subset K if and only if X is a regular space which is a compact-covering,[1] open image of a metric space.*

THEOREM 3. *Let X be a T_1 space. Then $(X \times I, \{(x, 0)\})$ is a semi-canonical pair for each point $x \in X$ if and only if X is a regular, first countable space.*

Throughout this paper, the following notations will be used: X_0 and X_n denote the subspaces $X \times \{0\}$ and $X \times \{1/n\}$ of $X \times I$ for $n = 1, 2, \ldots$; π denotes the projection from $X \times I$ onto X; and I_n denotes the subspace $[0, 1/n]$ of I for $n = 1, 2, \ldots$.

All spaces in this paper are T_1, and all maps are continuous.

2. Proof of Theorem 1. The sufficiency of the condition is clear, since every pair (X, A) in a metric space X is semi-canonical (cf. **[6]**). To prove necessity, suppose that there exists a semi-canonical cover $\mathcal{V}$ for $(X \times I, X_0)$. Put

$$\mathcal{U}_n = \pi(\mathcal{V}|X_n) \quad (= \{\pi(V \cap X_n) \colon V \in \mathcal{V}\})$$

for $n = 1, 2, \ldots$. Then $\{\mathcal{U}_n \colon n = 1, 2, \ldots\}$ is clearly a sequence of open covers of X.

Let us show that, for each point $x \in X$, the system $\{\mathrm{St}^2(x, \mathcal{U}_n) \colon n = 1, 2, \ldots\}$ forms a neighborhood base at x, where $\mathrm{St}^2(x, \mathcal{U})$ denotes the set $\mathrm{St}(\mathrm{St}(x, \mathcal{U}), \mathcal{U})$. Then X is metrizable by a theorem of K. Morita **[10]**. To complete the proof, let x be any point of X and G an arbitrary neighborhood of x in X. Since $\mathcal{V}$ is a semi-canonical cover for $(X \times I, X_0)$, there exist a neighborhood H_1 of x in X and a positive integer m such that $\mathrm{St}(H_1 \times I_m, \mathcal{V}) \subset G \times I$ holds. Again, for the neighborhood H_1 of x there exist a neighborhood H_2 of x in X and a positive integer n such that $n \geqslant m$ and $\mathrm{St}(H_2 \times I_n, \mathcal{V}) \subset H_1 \times I$. Now, let us show $\mathrm{St}^2(x, \mathcal{U}_n) \subset G$. Pick an arbitrary point y in $\mathrm{St}^2(x, \mathcal{U}_n)$. Then there are two members U, U' of $\mathcal{U}_n$ with $x \in U$, $y \in U'$ and $U \cap U' \neq \varnothing$. Let z be a point of $U \cap U'$. By the definition of $\mathcal{U}_n$ there exist V, V' in $\mathcal{V}$ such that

$$U = \pi(V \cap X_n) \quad \text{and} \quad U' = \pi(V' \cap X_n).$$

Hence, $(x, 1/n) \in V$, $(z, 1/n) \in V \cap V'$ and $(y, 1/n) \in V'$ hold. The first inclusion $(x, 1/n) \in V$ implies $V \subset H_1 \times I$, because $(x, 1/n)$ belongs to $H_2 \times I_n$; the second one implies $V' \subset G \times I$, because $(z, 1/n) \in V$ shows $(z, 1/n) \in H_1 \times I_n$ and $(z, 1/n) \in V'$ yields $V' \cap (H_1 \times I_n) \neq \varnothing$; and, as a consequence, the last inclusion $(y, 1/n) \in V'$ implies $y \in G$, which completes the proof.

3. Some lemmas.

LEMMA 1. *Let X be a space. If the pair $(X \times I, \{(x, 0)\})$ is semi-canonical for every point $x \in X$, then X is a regular space.*

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[1] A continuous map $f \colon X \to Y$ is called *compact-covering* if every compact subset of Y is the image of some compact subset of X.

PROOF. Using the same notations as in the proof of Theorem 1, it has been shown that, for a given point x of X and an arbitrary neighborhood G of x, there exists an open cover $\mathfrak{U}_n$ of X such that $\mathrm{St}^2(X, \mathfrak{U}_n) \subset G$ holds. Clearly, $\mathrm{St}(x, \mathfrak{U}_n)$ is a neighborhood of x, whose closure is contained in $\mathrm{St}^2(x, \mathfrak{U}_n)$ and hence in G. This proves that X is a regular space.

If $A \subset X$, then an *X-base for A* is a collection $\mathfrak{U}$ of open subsets of X such that, if $x \in A$ and V is a neighborhood of x in X, then $x \in U \subset V$ for some $U \in \mathfrak{U}$.

LEMMA 2. *Let X be a regular (T_1) space and K a compact subset of X. If there exists a countable X-base for K, then there exists an X-base $\bigcup_{n=1}^{\infty}\mathfrak{P}_n$ for K such that*

(1) *$\mathfrak{P}_n$ is a finite collection whose union covers K for $n = 1, 2, \ldots$,*

(2) *$\{\bar{P}: P \in \mathfrak{P}_{n+1}\}$ refines $\mathfrak{P}_n$ for $n = 1, 2, \ldots$, and*

(3) *for each point x of K and each neighborhood G of x in X, there exist a positive integer n and a neighborhood H of x in X such that* $\mathrm{St}(H, \mathfrak{P}_n) \subset G$.

PROOF. Let $\mathfrak{B}$ be the given countable X-base for K. Since $\mathfrak{B}|K$ is a countable base for K itself, K is metrizable. Hence, for any subset E of K, the diameter $\delta(E)$ of E is well defined and also, for any cover $\mathfrak{E}$ of K, the mesh $\mathfrak{E} = \sup\{\delta(E); E \in \mathfrak{E}\}$ is well defined.

For each n, let $\mathfrak{U}_n$ be a finite subcollection of $\mathfrak{B}$ such that

$(1)_n$ $\mathfrak{U}_n$ covers K, and

$(2)_n$ mesh $\mathfrak{U}_n|K < 1/2^n$.

Let $\{\mathfrak{V}_n: n = 1, 2, \ldots\}$ be the set of all finite subcollections of $\mathfrak{B}$, each of which forms a minimal cover with respect to K; that is, any proper subcollection of $\mathfrak{V}_n$ does not cover K for $n = 1, 2, \ldots$. Put $\mathfrak{W}_1 = \mathfrak{U}_1 \wedge \mathfrak{V}_1$ $(= \{U \cap V: U \in \mathfrak{U}_1, V \in \mathfrak{V}_1\})$ and $\mathfrak{W}_{n+1} = \mathfrak{W}_n \wedge \mathfrak{U}_{n+1} \wedge \mathfrak{V}_{n+1}$ for $n = 1, 2, \ldots$. Then each $\mathfrak{W}_n$ is a finite collection of open subsets of X whose union covers K.

Next, by induction on n, let us construct a finite collection $\mathfrak{F}_n$ of closed subsets of K, a finite collection $\mathfrak{P}_n$ of open subsets of X and a function φ_n from $\mathfrak{F}_n$ onto $\mathfrak{P}_n$ such that the following conditions are satisfied:

$(3)_n$ $\mathfrak{F}_n$ is a closed cover of K which refines $\mathfrak{W}_n \wedge \mathfrak{P}_{n-1}$, where $\mathfrak{P}_0 = \{X\}$,

$(4)_n$ $\mathfrak{P}_n$ refines $\mathfrak{W}_n \wedge \mathfrak{P}_{n-1}$,

$(5)_n$ if $F \in \mathfrak{F}_n$, then $F \subset \varphi_n(F)$,

$(6)_n$ if $F \in \mathfrak{F}_n$ and $F \subset 0 \in \bigcup_{i=1}^{n-1}\mathfrak{P}_i \cup \bigcup_{i=1}^{n}(\mathfrak{U}_i \cup \mathfrak{V}_i)$, then $\overline{\varphi_n(F)} \subset 0$, and

$(7)_n$ if $F \cap F' = \varnothing$, then $\varphi_n(F) \cap \varphi_n(F') = \varnothing$ for $F, F' \in \mathfrak{F}_n$.

Let $\mathfrak{W}_1 = \{W_1, \ldots, W_k\}$. Since $\mathfrak{W}_1$ covers K and K is normal, there exists a closed cover $\mathfrak{F}_1 = \{F_1, \ldots, F_k\}$ of K such that $F_i \subset W_i$ for $i = 1, \ldots, k$. Hence $\mathfrak{F}_1$ satisfies condition $(3)_1$. Since X is regular and $\mathfrak{F}_1$ is a finite collection, each member of which is compact, and since $\mathfrak{U}_1$ and $\mathfrak{V}_1$ are also finite collections, it is easy to see that the function φ_1 and $\mathfrak{P}_1 = \varphi_1(\mathfrak{F}_1)$ are well defined to satisfy conditions $(4)_1$–$(7)_1$, as well. The situation in each step is the same as above, and thus $\mathcal{F}_n$, φ_n and $\mathcal{P}_n$ are all constructed quite similarly.

Now, it remains to show that the sequence $\{\mathcal{P}_n: n = 1, 2, \ldots\}$ is the required one in Lemma 2. Since $\mathcal{P}_n$ is finite and satisfies $(3)_n$ and $(5)_n$, $\mathcal{P}_n$ satisfies the condition (1). By $(3)_n$ and $(6)_n$, $\mathcal{P}_n$ satisfies the condition (2). To prove that $\{\mathcal{P}_n: n = 1, 2, \ldots\}$ satisfies the condition (3), let x be any point of K and G an arbitrary neighborhood of x in X. Since $\mathcal{B}$ is an X-base for K, there exists a $B_0 \in \mathcal{B}$ such that $x \in B_0 \subset G$.[2] Let $\mathcal{V}$ be a finite subcollection of $\mathcal{B}$ which is a minimal cover with respect to K and which keeps B_0 as the only member of $\mathcal{V}$ containing x. Since K is a compact T_2 space and since $\mathcal{B}$ is an X-base for K, such $\mathcal{V}$ certainly exists; further, for some n, $\mathcal{V} = \mathcal{V}_n$.

Let $F_0 \in \mathcal{F}_n$ be a member with $x \in F_0$. Then $F_0 \subset B_0$ holds, because $\mathcal{F}_n$ refines $\mathcal{W}_n$ which refines $\mathcal{V}_n$ and B_0 is the only member of $\mathcal{V}$ containing x; and also, by $(5)_n$ and $(6)_n$, the inclusions $F_0 \subset \varphi_n(F_0) \subset B_0$ hold. Since $\varphi_n(F_0)$ is an open set containing x, there exists a postive integer m such that $m \geqslant n$ and $d(x, K - \varphi_n(F_0)) > 1/2^m$, where d denotes the metric function on K. Since $\mathcal{F}_{m+1}$ is a cover of K by $(3)_{m+1}$, there exists an $F_1 \in \mathcal{F}_{m+1}$ containing x. To complete the proof, it suffices to show that

$$\mathrm{St}(\varphi_{m+1}(F_1), \mathcal{P}_{m+1}) \subset \varphi_n(F_0),$$

because $\varphi_{m+1}(F_1)$ is an open set in X containing x and $\varphi_n(F_0)$ is contained in B_0, which is contained in G. Let P be an arbitrary member of $\mathcal{P}_{m+1}$ and F the corresponding member of $\mathcal{F}_{m+1}$ by $P = \varphi_{m+1}(F)$. If $P \cap \varphi_{m+1}(F_1) \neq \varnothing$, then by $(7)_{m+1}$, $F \cap F_1 \neq \varnothing$. Since $\mathcal{F}_{m+1}$ refines $\mathcal{P}_{m+1}$ by $(5)_{m+1}$ and $\mathcal{P}_{m+1}$ refines $\mathcal{W}_{m+1}$ by $(4)_{m+1}$, and since $\mathcal{W}_{m+1}$ refines $\mathcal{U}_{m+1}$ whose mesh restricting to K is less than $1/2^{m+1}$, the diameter $\delta(F \cup F_1)$ is less than $1/2^m$. Since x belongs to F_1, by the choice of m, $F \cup F_1 \subset \varphi_n(F_0)$ holds. Again by $(6)_{m+1}$, $\varphi_{m+1}(F) \subset \varphi_n(F_0)$ and thus $P \subset \varphi_n(F_0)$ holds, which completes the proof.

LEMMA 3. *Let X be a regular (T_1) space and K a compact subset of X. If there exists a countable X-base for K, then (X, K) is a semi-canonical pair.*

PROOF. Let $\bigcup_{n=1}^{\infty}\mathcal{P}_n$ be an X-base for K obtained by Lemma 2. For each n, put $G_n = \bigcup\{P: P \in \mathcal{P}_n\}$. Then, by conditions (1) and (2) in Lemma 2, $\overline{G}_{n+1} \subset G_n$ for $n = 1, 2, \ldots$ and $K \subset \bigcap_{n=1}^{\infty} G_n$, and by condition (3) and by the fact that K is compact, it is easily seen that $K = \bigcap_{n=1}^{\infty} G_n$.

Now, put $\mathcal{V}_0 = \{X - \overline{G}_2\}$ and $\mathcal{V}_n = \mathcal{P}_n|(G_n - \overline{G}_{n+2})$ for $n = 1, 2, \ldots,$ and put $\mathcal{V} = \bigcup_{n=0}^{\infty}\mathcal{V}_n$. Then it will be shown that $\mathcal{V}$ is a semi-canonical cover for (X, K). Clearly, $\mathcal{V}$ is an open cover of $X - K$. To complete the proof, let x be any point of K and U an arbitrary neighborhood of x in X. By condition (3) in Lemma 2, there exist a positive integer n and a neighborhood H of x in X such that $\mathrm{St}(H, \mathcal{P}_n) \subset U$. Put $W = H \cap G_{n+1}$. Then W is a neighborhood of x in X such that $W \cap V = \varnothing$ for each $V \in \bigcup_{i=1}^{n-1}\mathcal{V}_i$.

[2] If K is singleton, then $\bigcup_{n=1}^{\infty}\mathcal{P}_n$ is easily chosen from the given countable X-base for K, because X is regular. So, assuming that K is not a singleton, B_0 is picked out from $\mathcal{B}$ such that $K - B_0 \neq \varnothing$.

Therefore

$$\operatorname{St}(W, \mathcal{V}) = \operatorname{St}\left(W, \bigcup_{i>n} \mathcal{V}_i\right) \subset \operatorname{St}\left(W, \bigcup_{i>n} \mathcal{P}_i\right) \subset \operatorname{St}(H, \mathcal{P}_n) \subset U$$

by condition (2) in Lemma 2, and that completes the proof.

4. Proofs of Theorems 2 and 3. The following characterization of the compact-covering open images of metric spaces, due to E. Michael and K. Nagami [9] will be used in the proof of Theorem 2.

THEOREM M-N (E. MICHAEL AND K. NAGAMI).[3] *For a T_2 space X, the following conditions are equivalent*:

(1) *X is the compact-covering open image of a metric space.*

(2) *Every compact subset of X is metrizable and of countable character in X.*[4]

(3) *Every compact subset of X has a countable X-base.*

PROOF OF THEOREM 2. *Necessity*. Let $(X \times I, K \times \{0\})$ be a semi-canonical pair for any compact subset K of X. Then X is a regular space by Lemma 1 putting K in the assumption a singleton. Next, it will be shown that each compact subset K of X has a countable X-base. Then X is the compact-covering open image of a metric space by Theorem M-N.

To complete the proof, let K be a compact subset of X. By the assumption, there exists a semi-canonical cover $\mathcal{V}$ for $(X \times I, K \times \{0\})$. Put $\mathcal{V}_n$ the finite subcollection of $\mathcal{V}$ which covers $K \times \{1/n\}$, and put $\mathcal{U}_n = \pi(\mathcal{V}_n | X_n)$ for $n = 1, 2, \ldots$.

Then it is easy to show that the collection $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ is the required X-base for K, by the same technique as in the proof of Theorem 1.

Sufficiency. It is easy to check that, if X is the compact-covering open image of a metric space, then so is $X \times I$. Hence, for any compact subset K of X, $K \times \{0\}$ has a countable $X \times I$-base by Theorem M-N, and thus $(X \times I, K \times \{0\})$ is a semi-canonical pair by Lemma 3, which completes the proof.

PROOF OF THEOREM 3. *Necessity*. By Lemma 1, X is a regular space. The first countability of X is proved by the same technique as in the proof of the necessity in Theorem 2, replacing K by a singleton.

Sufficiency. If X is a regular (T_1) first countable space, then so is $X \times I$. In general, it is easily seen that, in any regular (T_1) first countable space Y, the pair $(Y, \{y\})$ is always semi-canonical for each point $y \in Y$. This completes the proof.

5. Comments. 1. From the proofs of Theorems 1, 2 and 3, it is easy to see that, in the conditions of these theorems, the closed interval I may be

[3] The fact (2) → (3) was proved by M. M. Čoban [5]; for completely regular space X, it had previously been obtained by A. V. Arhangel'skiĭ [1].

[4] A set $K \subset X$ is *of countable character* in X if there is a *countable outer base* $\{U_n: n = 1, 2, \ldots\}$ at K in X (i.e. each U_n is open and contains K, and every open set containing K contains some U_n) (cf. [9]).

replaced by any space containing a convergent sequence. By such replacement in Theorem 1, one obtains a slight modification of the proof of the following theorem due to D. M. Hyman [7], remembering two facts: (1) The closed image of a metric space is a Fréchet-Urysohn space (cf. [8]); and (2) any pair (X, A) is semi-canonical if X is the closed image of a metric space (cf. [7]).

THEOREM (D. HYMAN). *If X and Y are nondiscrete spaces and if $X \times Y$ is the closed image of a metric space, then X and Y are metrizable.*

2. The semi-canonical property need not be two-productive. For example, let $X = N \cup \{p\}$ be a subspace of Stone-Čech compactification βN of N $(= \{1, 2, \ldots\})$ with $p \in \beta N - N$. Then it is well known that X is not first countable at p, and thus $(X \times I, \{(p, 0)\})$ is not semi-canonical by Theorem 3. However, it is easy to see that any pair (X, A) is always semi-canonical.

This example also shows that, in the conditions of Theorems 1 and 2, $X \times I$ cannot be replaced by X. Clearly, then, the semi-canonical property in X is very different from the semi-canonical property in $X \times I$.

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