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A GENERALIZED KLEENE-MOSCHOVAKIS THEOREM

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ABSTRACT. Moschovakis generalized a theorem of Kleene to prove that if $\mathfrak{X}$ is a collection of subsets of any acceptable structure $\mathfrak{M}$ such that $(\mathfrak{M}, \mathfrak{X}) \models \Delta_1^1$ comprehension, every hyperelementary subset of $\mathfrak{M}$ is in $\mathfrak{X}$. We prove an analogous result for arbitrary $\mathfrak{M}$. We also get analogous results for $\mathfrak{M}$ with an extra quantifier Q.

In **[74]** Moschovakis generalizes a theorem of Kleene to prove that over any acceptable structure $\mathfrak{M}$, the smallest nonempty set $\mathfrak{X}$ of subsets of $\mathfrak{M}$ such that $\langle \mathfrak{M}, \mathfrak{X} \rangle \models \Delta_1^1$ comprehension is the set of hyperelementary subsets of $\mathfrak{M}$. Since then, interest in generalizations has arisen. First, in light of recent work with extra quantifiers, Moschovakis has asked whether the same result holds with an extra monotone quantifier Q. (This we show to be false.) Second, recursion-theoretic work over nonacceptable structures has now aroused interest. (See, e.g., Barwise **[75]**.) In this spirit we eliminate most of the coding machinery from the proof.

We follow generally notation from Moschovakis **[74]** and Barwise **[75]**, **[76]**. We assume fixed finite language $\mathfrak{L}$ for a structure $\mathfrak{M}$; we consider sentences both with and without the Q quantifier, a monotone quantifier on the structure $\mathfrak{M}$. We use capital letters for second order variables (ranging over elements of $\mathfrak{X}$). By a first order (i.e. $\mathfrak{L}_{\omega\omega}$ or $\mathfrak{L}(Q)_{\omega\omega}$ formula) we mean a formula with no second order quantifiers–we specifically allow second order parameters. We let $\breve{Q}x$ stand for $\neg Qx \neg$.

DEFINITION. A formula Φ is Σ_1^1 (resp. $\Sigma_1^1(Q)$) if it is of the form $\exists X_1 \cdots \exists X_n \varphi(X_1 \cdots X_n \vec{Y}\vec{y})$ for φ in $\mathfrak{L}_{\omega\omega}$ (resp. $\mathfrak{L}(Q)_{\omega\omega}$). Φ is essentially $\Sigma_1^1(Q)$ if it is in the smallest class of formulas containing the $\mathfrak{L}(Q)_{\omega\omega}$ formulas and closed under $\wedge$, $\vee$, $\exists x$, $\forall x$, Qx, $\breve{Q}x$, and $\exists X$.

DEFINITION. Let $\mathfrak{M}$ be a structure and let $\mathfrak{X}$ be a collection of subsets of $\mathfrak{M}$ (or $\mathfrak{M}^n$). We say $Y \subseteq \mathfrak{M}$ (or $\mathfrak{M}^n$) is Δ_1^1 definable over $\langle \mathfrak{M}, \mathfrak{X} \rangle$ if both Y and $\mathfrak{M} - Y$ (or $\mathfrak{M}^n - Y$) are definable by Σ_1^1 formulas (with parameters from $\mathfrak{M}$ and $\mathfrak{X}$). We say $\langle \mathfrak{M}, \mathfrak{X} \rangle \models \Delta_1^1\text{CA}$ (Δ_1^1 Comprehension Axiom) if every subset

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of $\mathfrak{M}$ (or $\mathfrak{M}^n$) which is Δ^1_1 definable over $\langle \mathfrak{M}, \mathfrak{X} \rangle$ is an element of $\mathfrak{X}$. We similarly define $\Delta^1_1(Q)$ definable, essentially $\Delta^1_1(Q)$ definable, etc.

THEOREM 1 [BARWISE-GANDY-MOSCHOVAKIS]. *Let $\mathfrak{M}$ be a structure and let $\mathfrak{X}$ be the collection of subsets of $\mathfrak{M}$ (or $\mathfrak{M}^n$) in the smallest admissible set above $\mathfrak{M}$ (i.e., the smallest admissible set built upon the set $\mathfrak{M}$ of urelements and containing $\mathfrak{M}$ as an element–Barwise's* $\mathrm{HYP}_{\mathfrak{M}}$). *Then* $\langle \mathfrak{M}, \mathfrak{X} \rangle \models \Delta^1_1$ CA.

PROOF. Fairly obvious–done in great detail in Barwise and Schlipf [**76**]. □

DEFINITION. We say an admissible set is *weakly Q-admissible* if it satisfies the Δ_0-separation and Δ_0-collection axioms for Δ_0 formulas containing Q's (i.e., bounded Q quantifiers). (Note that in our context all Q's and $\check{Q}$'s are bounded since the Q is a quantifier on $\mathfrak{M}$ and $\mathfrak{M}$ will be an element of each of our admissible sets.) We say the admissible set is *strongly Q-admissible* if, in addition, it satisfies Q collection:

$$Qx \in \mathfrak{M} \exists y \varphi(x, y, z) \Rightarrow \exists b\ Qx \in \mathfrak{M} \exists y \in b \varphi(x, y, z),$$
$$\check{Q}x \in \mathfrak{M} \exists y \varphi(x, y, z) \Rightarrow \exists b\ \check{Q}x \in \mathfrak{M} \exists y \in b \varphi(x, y, z)$$

for φ a Δ_0 formula–i.e., containing only bounded quantifiers (including Q's).

DEFINITION. Let $\mathrm{HYP}_{\mathfrak{M}}$, $\mathrm{HYP}_{\mathfrak{M},q}$ and $\mathrm{HYP}^{\#}_{\mathfrak{M},q}$ denote the smallest admissible set above $\mathfrak{M}$, the smallest weakly Q-admissible set above $\mathfrak{M}$, and the smallest strongly Q-admissible set above $\mathfrak{M}$, respectively. Many cases are known where $\mathrm{HYP}_{\mathfrak{M},q} \neq \mathrm{HYP}^{\#}_{\mathfrak{M},q}$.

THEOREM 2 (OBVIOUS GENERALIZATION). *Let $\mathfrak{M}$ be a structure and let $\mathfrak{X}$ be the collection of subsets of $\mathfrak{M}$ (or $\mathfrak{M}^n$) which are in* $\mathrm{HYP}_{\mathfrak{M},q}$. *Then* $\langle \mathfrak{M}, \mathfrak{X} \rangle \models \Delta^1_1(Q)$CA. *Similarly, if $\mathfrak{X}$ is the set of subsets of $\mathfrak{M}$ (or $\mathfrak{M}^n$) in* $\mathrm{HYP}^{\#}_{\mathfrak{M},q}$, *then* $\langle \mathfrak{M}, \mathfrak{X} \rangle \models$ *essential* $\Delta^1_1(Q)$CA.

In the cases where $\mathfrak{M}$ has an inductive pairing function or a deterministic-Q-inductive pairing function (which we define below), or a Q-inductive pairing function, we get full converses to Theorems 1 and 2. We prove the converses following the basic idea of Moschovakis–showing that each stage of each inductive definition, except possibly the last stage, is in the appropriate $\mathfrak{X}$.

We define (positive, elementary) *inductive* and Q-(positive, elementary)-inductive as usual. A formula $\varphi(x, R_+)$ is *deterministic* if it is in the smallest class of formulas containing the atomic formulas and closed under $\wedge, \vee, \neg$, $\exists x$, and $\forall x$, and under $Q'x$ and $\check{Q}'x$ where

$$Q'x(\varphi, \psi) \text{ is } \exists x(\varphi \wedge \psi) \vee [\forall x(\varphi \vee \psi) \wedge Qx\varphi],$$
$$\check{Q}'x(\varphi, \psi) \text{ is } \exists x(\varphi \wedge \psi) \vee [\forall x(\varphi \vee \psi) \wedge \check{Q}x\varphi].$$

(Here we follow Barwise [**76**] except in notation.)

DEFINITION. Let $\mathrm{HE}_{\mathfrak{M}}$, $\mathrm{HE}_{\mathfrak{M},q}$, and $\mathrm{HE}^{\#}_{\mathfrak{M},q}$ denote the set of hyperelementary (i.e., inductive and coinductive), deterministic-Q-hyperelementary (i.e., inductive and coinductive from inductive definitions given by deterministic-Q-formulas), and Q-hyperelementary relations on $\mathfrak{M}$. Let $\mathrm{Rel}(\mathfrak{M})$ be the set of all relations on $\mathfrak{M}$.

THEOREM 3 [*After Barwise-Gandy-Moschovakis; due in this generality to Barwise*].

$$\mathrm{HE}_{\mathfrak{M}} \subseteq \mathrm{HYP}_{\mathfrak{M}} \cap \mathrm{Rel}(\mathfrak{M}); \quad \mathrm{HE}_{\mathfrak{M},q} \subseteq \mathrm{HYP}_{\mathfrak{M},q} \cap \mathrm{Rel}(\mathfrak{M});$$
$$\mathrm{HE}^{\#}_{\mathfrak{M},q} \subseteq \mathrm{HYP}^{\#}_{\mathfrak{M},q} \cap \mathrm{Rel}(\mathfrak{M}).$$

(Thus the same conclusion holds if we restrict ourselves to a fixed arity.) If $\mathfrak{M}$ has an inductive pairing function, a deterministic-Q-inductive pairing function, or a Q-inductive pairing function, respectively, the three inclusions become equalities.

PROOF. See Barwise [75], [76]. □

DEFINITION. A formula $\varphi(x, R_+)$ is ID (in *inductive-disjunctive form*) if it is of the form $\varphi_1 \vee \cdots \vee \varphi_n$, where each of the φ_i's is of the form $\exists y\psi_i$, $\forall y\psi_i$, or ψ_i, where ψ_i is quantifier free. A formula is dQ-ID (in *deterministic-Q-inductive-disjunctive form*) if we also allow the φ_i's to be of the form $Q'y(\psi_i, \chi_i)$ and $\breve{Q}'y(\psi_i, \chi_i)$ for ψ_i, χ_i quantifier free.

LEMMA 4 [WELL KNOWN]. *For every $\varphi(x_1, \ldots, x_k\ R_+)$ of $\mathfrak{L}(R)_{\omega\omega}$ there is an ID $\psi(x_1 \cdots x_n\ S_+)$ of $\mathfrak{L}(S)_{\omega\omega}$ such that the fixed point I_φ is a section of I_ψ. For every deterministic $\varphi(\vec{x}, R_+)$ of $\mathfrak{L}(R)(Q)_{\omega\omega}$ there is a dQ-ID $\psi(\vec{x}, R_+)$ such that I_φ is a section of I_ψ.*

DEFINITION. A pseudohierarchy for a formula $\varphi(x_1 \cdots x_k\ R_+)$ on a structure $\mathfrak{M}$ is a subset H of $\mathfrak{M}^{2k}$ with the following properties, where $\vec{x} \leqslant \vec{y}$ abbreviates $(x_1 \cdots x_k\, y_1 \cdots y_k) \in H$; $\vec{x} \sim \vec{y}$ abbreviates $\vec{x} \leqslant \vec{y} \wedge \vec{y} \leqslant \vec{x}$; and $\vec{x} < \vec{y}$ abbreviates $\vec{x} \leqslant \vec{y} \wedge \vec{y} \nleqslant \vec{x}$:

(1) $\leqslant$ is a (nonstrict) partial ordering of the field of $\leqslant$ in which all elements are comparable.

(2) $\sim$ is an equivalence relation on its field.

(3) $\forall \vec{y} \in \mathrm{field}(\leqslant)\ \forall \vec{x}((\vec{x} \leqslant \vec{y}) \Leftrightarrow \varphi(\vec{x}, \{\vec{z}: \vec{z} < \vec{y}\}))$.

Intuitively, a pseudohierarchy is supposed to code–under nice circumstances–an initial segment of the inductive construction of I_φ. Clearly, if the $\leqslant$ relation of the pseudohierarchy is well founded, that is the case. But more importantly, we have the following result:

LEMMA 5. *Let H be a pseudohierarchy for φ on $\mathfrak{M}$. If the relation $\leqslant$ determined by H is not well founded, then for any y in the nonwellfounded part, and for any $\vec{z} \in I_\varphi$, $\vec{z} \leqslant \vec{y}$.*

PROOF. By induction $|\vec{z}|_\varphi$. For successor levels "back up" to an earlier level in the nonwellfounded part. (Phocion Kolaitis has noted that we do not even need to assume that $\leqslant$ is transitive for this result.) □

THEOREM 6. *Let X be a collection of $2k$-ary relations on $\mathfrak{M}$; let $\langle \mathfrak{M}, \mathfrak{X} \rangle \models$*

essential $\Delta_1^1(Q)$ CA; *and let* $\varphi(x_1 \cdots x_k\ R_+) \in \mathfrak{L}(R)(Q)_{\omega\omega}$. *Let* $\vec{z} \in I_\varphi$. *Then* $\{(\vec{x}, \vec{y})\colon |\vec{x}|_\varphi < |\vec{y}|_\varphi < |\vec{z}|_\varphi\} \in \mathfrak{X}$.

Proof. We prove this by induction on $|\vec{z}|_\varphi$, following the general outline of Moschovakis' proof, but using pseudohierarchies to avoid the second stage comparison theorem. (Alternatively, we could use Aczel's proof of that theorem.) The result is trivial if $|\vec{z}|_\varphi$ is 0 or a successor ordinal. Let $\lambda = |\vec{z}|_\varphi$ be a limit. Clearly $|\vec{x}|_\varphi \leqslant |\vec{y}|_\varphi < |\vec{z}|_\varphi$ iff $\langle \mathfrak{M}, \mathfrak{X}\rangle \models \exists H$ (H is a pseudohierarchy for φ and $(\vec{x}, \vec{y}) \in H$ and $\vec{z} \notin$ field(H)). Thus we need only find an essentially Σ_1^1 definition of the complement of this relation. That definition is clear once we show that $\{\vec{x}\colon |\vec{x}|_\varphi \geqslant |\vec{z}|_\varphi\} \in \mathfrak{X}$. The complement of this set is clearly Σ_1^1 by the same argument as before. So we need but show the set is essentially Σ_1^1.

$$
\begin{aligned}
|\vec{x}|_\varphi \geqslant |\vec{z}|_\varphi &\Leftrightarrow \varphi\big(\vec{z}, \{\vec{y}\colon |\vec{y}|_\varphi < |\vec{x}|_\varphi\}\big)\\
&\Leftrightarrow \varphi\big(\vec{z}, \{\vec{y}\colon |\vec{y}|_\varphi < |\vec{x}|_\varphi \wedge |\vec{y}|_\varphi < |\vec{z}|_\varphi\}\big)\\
&\Leftrightarrow \varphi\big(\vec{z}, \{\vec{y}\colon \exists H \in \mathfrak{X}\ (H \text{ is a pseudohierarchy}\\
&\qquad \text{for } \varphi \text{ and } \vec{x}, \vec{z} \notin \text{field}(H) \text{ and } \vec{y} \in \text{field}(H))\}\big),
\end{aligned}
$$

and the last formula, interpreted over $\langle \mathfrak{M}, \mathfrak{X}\rangle$, is obviously essentially Σ_1^1 since φ is R-positive. □

Theorem 7. *Let* $\mathfrak{X}$ *be a collection of* $2k$*-ary relations on* $\mathfrak{M}$; *let* $\langle \mathfrak{M}, \mathfrak{X}\rangle \models \Delta_1^1$CA (*respectively*, $\Delta_1^1(Q)$CA); *and let* $\varphi(x_1 \cdots x_k\ R_+)$ *be ID* (*resp. dQ-ID*). *Let* $\vec{z} \in I_\varphi$. *Then* $\{(x, y)\colon |\vec{x}|_\varphi \leqslant |\vec{y}|_\varphi < |\vec{z}|_\varphi\} \in \mathfrak{X}$.

Proof. Exactly as for Theorem 6, save that we need to get a Σ_1^1 definition of $\{\vec{x}\colon |\vec{x}|_\varphi \geqslant |\vec{z}|_\varphi\}$ instead of an essentially Σ_1^1 definition. We use the normal form to carry out the same sort of trick Moschovakis did with the coding. We have that $\varphi(\vec{z}, \{\vec{y}\colon |\vec{y}|_\varphi < |\vec{z}|_\varphi\})$—thus that $\langle \mathfrak{M}, \mathfrak{X}\rangle \models \varphi(\vec{z}, \{\vec{y}\colon \exists H(H$ is a pseudohierarchy for φ and $\vec{y} \in$ field(H) and $\vec{z} \notin$ field$(H))\})$. And we assumed that $\varphi(\vec{z}, R_+)$ is of the form

$$\bigvee_{i<k} \exists p\psi_i(\vec{z}, p, R_+) \vee \bigvee_{j<l} \forall p\psi_j(\vec{z}, p, R_+).$$

Now $I^{<\lambda} = \bigcup_{\beta<\lambda} I^\beta$. One of the existential disjuncts could not hold at $\vec{z}, I^{<\lambda}$, since it would then hold at some $\vec{z}, I^\beta$ for $\beta < \lambda$, contradicting the fact that $|\vec{z}|_\varphi = \lambda$. So for suitable ψ_j,

$$\forall p\psi_j(\vec{z}, p, I^{<\beta})$$

holds first for $\beta = \lambda$. Define $(p, \vec{y})$ to be in Z iff

$$
\begin{aligned}
\langle \mathfrak{M}, \mathfrak{X}\rangle \models \exists H\ \big(&H \text{ is a pseudohierarchy for } \varphi \text{ and } \vec{z} \notin \text{field}(H)\\
&\text{and } \vec{y} \in \text{field}(H) \text{ and } \psi_j(\vec{z}, p, \text{field}(H)),\\
&\text{and this last fails for all proper initial segments of } H\\
&\qquad (\text{under the } \leqslant \text{ relation defined before})\big)
\end{aligned}
$$

iff

$$\langle \mathfrak{M}, \mathfrak{X} \rangle \models \forall H \big(H \text{ is a pseudohierarchy for } \varphi \text{ and } \vec{z} \notin \text{field}(H) \text{ and } \psi_j(\vec{z}, p, \text{field}(H)) \Rightarrow \vec{y} \in \text{field}(H) \big).$$

So $Z \in \mathfrak{X}$. But then $|\vec{x}|_\varphi \geqslant |\vec{z}|_\varphi \Leftrightarrow \forall p((p, \vec{x}) \notin Z)$. □

Clearly, if $\mathfrak{M}$ has a definable pairing function we can code $\mathfrak{M}^n$ upon $\mathfrak{M}$ elementarily. And if $\mathfrak{M}$ has a hyperelementary pairing function we can do such coding inside any collection of $2k$-ary relations $\mathfrak{X}$ such that $\langle \mathfrak{M}, \mathfrak{X} \rangle \models \Delta_1^1\text{CA}$ if the pairing function is defined inductively by a k-variable ID formula. Similar remarks hold for dQ-ID and $\mathfrak{L}(Q)_{\omega\omega}$ defining formulas. Thus, combining previous results, we get:

THEOREM 8. *Let $\mathfrak{M}$ have a definable pairing function. Then the set of hyperelementary subsets of $\mathfrak{M}$ (resp. deterministic-Q-hyperelementary or Q-hyperelementary subsets of $\mathfrak{M}$) is the smallest collection $\mathfrak{X}$ of subsets of $\mathfrak{M}$ such* $\langle \mathfrak{M}, \mathfrak{X} \rangle \models \Delta_1^1\text{CA}$ (*resp.* $\Delta_1^1(Q)\text{CA}$ *or* essential $\Delta_1^1(Q)\text{CA}$).

THEOREM 9. *Let $\mathfrak{M}$ have an inductive pairing function given by the ID (resp. dQ-ID or $\mathfrak{L}(Q)_{\omega\omega}$) formula $\varphi(x_1 \cdots x_n\ R_+)$. Then the set of hyperelementary (resp. deterministic-Q-hyperelementary or Q-hyperelementary) $2n$-ary relations on $\mathfrak{M}$ is the smallest collection $\mathfrak{X}$ of $2n$-ary relations on $\mathfrak{M}$ such that* $\langle \mathfrak{M}, \mathfrak{X} \rangle \models \Delta_1^1\text{CA}$ (*resp.* $\Delta_1^1(Q)\text{CA}$, *or* essential $\Delta_1^1(Q)\text{CA}$).

OPEN PROBLEM. We have not identified the smallest $\mathfrak{X}$ such that $\langle \mathfrak{M}, \mathfrak{X} \rangle \models \Delta_1^1\text{CA}$ in the case that $\mathfrak{M}$ does not have a pairing function, nor, say, in the case that $\mathfrak{X}$ is a collection of subsets of $\mathfrak{M}$ and $\mathfrak{M}$ has only an inductive pairing function. We have only given bounds upon what this $\mathfrak{X}$ may be. And the analogous gap appears in our knowledge of $\Delta_1^1(Q)\text{CA}$ and essential $\Delta_1^1(Q)\text{CA}$.

REFERENCES

75 K. Jon Barwise, *Admissible sets and structures*, Springer-Verlag, Berlin and New York.

76 *Admissible sets and monotone quantifiers*, Lecture notes, U.C.L.A., spring 1976 (unpublished).

71 K. J. Barwise, R. O. Gandy and Y. N. Moschovakis, *The next admissible set*, J. Symbolic Logic **36** (1971), 108–120.

76 K. Jon Barwise and John Schlipf, *On recursively saturated models of arithmetic*, Model Theory and Algebra: A Memorial Tribute to Abraham Robinson, D. H. Saracino and B. Weispfenning (Editors), Lecture Notes in Math., vol. 498, Springer-Verlag, Berlin, pp. 42–54.

74 Y. N. Moschovakis, *Elementary induction on abstract structures*, North-Holland, Amsterdam.

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