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MAXIMAL SUBGROUPS OF PRIME INDEX IN A FINITE SOLVABLE GROUP

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ABSTRACT. In this note we show that a maximal subgroup of a finite solvable group has prime index if and only if it admits a cyclic supplement which permutes with one of its Sylow systems. In particular, a finite solvable group is supersolvable if and only if each maximal subgroup admits a cyclic supplement which permutes with a Sylow system of the maximal subgroup.

System quasinormalizers were introduced in [3] as a generalization of P. Hall's system normalizers. A *system quasinormalizer* of a finite solvable group G is defined to be the subgroup generated by all cyclic subgroups of G which permute with each element of a fixed Sylow system of G. In this note we use system quasinormalizers to characterize the maximal subgroups of prime index in a finite solvable group. As a consequence of this characterization, a sharpening of a result by Kegel is achieved. More specifically, Kegel [2] has shown that the class of finite groups which have the property that each maximal subgroup admits a cyclic supplement of prime power order is slightly larger than the class of all finite supersolvable groups. It will be shown here that the class of finite supersolvable groups is precisely the class of all finite solvable groups with the property that each maximal subgroup admits a cyclic supplement which permutes with each element in a Sylow system of the maximal subgroup. In this note we will be concerned only with finite solvable groups.

For a finite solvable group G and a set of primes π, G_π denotes a Hall π-subgroup of G while G^π denotes a Hall π-complement of G. In the case when π consists of a single prime p, we write simply G_p and G^p. By a *Sylow system* of G is meant a complete set of permuting Hall subgroups of the group G. Let Σ denote a Sylow system of the solvable group G. If H is a subgroup of G and $\{G_\pi \cap H\colon G_\pi \in \Sigma\}$ is a Sylow system of H, we say Σ reduces into H and denote this Sylow system of H by $\Sigma \cap H$.

Let $\mathfrak{S}$ be a Sylow system of a subgroup H of the group G. The *system quasinormalizer* of $\mathfrak{S}$ in G, denoted $N_G^*(\mathfrak{S})$, is the subgroup of G generated by all cyclic subgroups of G which permute with each element of $\mathfrak{S}$. For an element x, if $\langle x\rangle$ permutes with each element of $\mathfrak{S}$, we will say that x is $\mathfrak{S}$-quasinormal in G.

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LEMMA. *Let G be a finite solvable group and M a maximal subgroup of G with prime index p. If $\mathfrak{S}$ is a Sylow system of M and Σ is a Sylow system of G with $\Sigma \cap M = \mathfrak{S}$, then $N_G^*(\Sigma) \subseteq N_G^*(\mathfrak{S})$.*

PROOF. In [3] it is observed that $N_G^*(\Sigma)$ is generated by the elements of prime power order which are Σ-quasinormal. It suffices then to show that all such elements lie in $N_G^*(\mathfrak{S})$. Let y be a Σ-quasinormal element of G with $|y| = r^\alpha$ for some prime r.

Suppose $r \neq p$. Let G^p be the Sylow p-complement of G in Σ. As $\langle y \rangle$ permutes with G^p, $y \in G^p$. In that $\Sigma \cap M = \mathfrak{S}$, $G^p = M^p \in \mathfrak{S}$ and $y \in M$. For $M_\pi \in \mathfrak{S}$ we have $G_\pi \cap M = M_\pi$ where G_π lies in Σ. $\langle y \rangle G_\pi$ is a group, and by the Dedekind identity, $\langle y \rangle G_\pi \cap M = \langle y \rangle (G_\pi \cap M) = \langle y \rangle M_\pi$. Hence $\langle y \rangle$ permutes with each M_π in $\mathfrak{S}$ so that $y \in N_G^*(\mathfrak{S})$. We may now assume $r = p$.

Let M_q be a Sylow q-subgroup in $\mathfrak{S}$ and G_q a Sylow q-subgroup in Σ so that $G_q \cap M = M_q$. For $q \neq p$, $M_q = G_q$, and since y is Σ-quasinormal, $\langle y \rangle$ permutes with M_q. For $p = q$, $|G_p : M_p| = |G: M| = p$ so that M_p is a normal subgroup of G_p. As $\langle y \rangle$ is a p-group and permutes with G_p, y must lie in G_p. That is, $y \in N_G(M_p)$ so that $\langle y \rangle$ also permutes with M_p. Thus $\langle y \rangle$ permutes with each Sylow subgroup of $\mathfrak{S}$ and so must permute with every element in $\mathfrak{S}$. Therefore in this case also $y \in N_G^*(\mathfrak{S})$ and the lemma is established.

THEOREM. *Let G be a finite solvable group and M a maximal subgroup of G with Sylow system $\mathfrak{S}$. M has a prime index in G if and only if $G = MN_G^*(\mathfrak{S})$.*

PROOF. Suppose first that M has prime index p. Let Σ be a Sylow system of G with $\Sigma \cap M = \mathfrak{S}$. M must complement a chief factor of G which has prime order p and $N_G^*(\Sigma)$ covers this chief factor [3, Corollary 2.3]. Hence $G = MN_G^*(\Sigma)$, so that by the preceding lemma the result follows.

On the other hand suppose $G = MN_G^*(\mathfrak{S})$ and let G be the minimal counterexample to the theorem. We may thus assume that M is core free. As $G = MN_G^*(\mathfrak{S})$, there are $\mathfrak{S}$-quasinormal elements of G which do not lie in M. Let y be an element of smallest possible order having this property. By the maximality of M, $G = M\langle y \rangle$. Thus $(\langle y \rangle \cap M)^G \subseteq M$, and since M is core free, $\langle y \rangle \cap M = \{1\}$. Thus $|y| = |G : M| = p^\alpha$ for some prime p.

Let M_p be the Sylow p-subgroup of M in $\mathfrak{S}$. Since $\langle y \rangle$ permutes with M_p, $G_p = M_p\langle y \rangle$ is a Sylow subgroup of G. Let T be a maximal subgroup of G_p with $T \supseteq M_p$. Since $T = M_p(\langle y \rangle \cap T)$, it follows that $p = |G_p : T| = |\langle y \rangle : \langle y \rangle \cap T|$ and $\langle y \rangle \cap T = \langle y^p \rangle$. Hence $T = M_p\langle y^p \rangle$ and $\langle y^p \rangle$ permutes with M_p.

For M_q a Sylow q-subgroup in $\mathfrak{S}$ with $p \neq q$, $M_q\langle y \rangle$ is p-supersolvable. If L is a maximal subgroup of $M_q\langle y \rangle$ containing M_q, then $L = M_q(\langle y \rangle \cap L)$. By the p-supersolvability of $M_q\langle y \rangle$, $p = |M_q\langle y \rangle : L| = |\langle y \rangle : \langle y \rangle \cap L|$ so that $\langle y \rangle \cap L = \langle y^p \rangle$. Thus $L = M_q\langle y^p \rangle$ and $\langle y^p \rangle$ permutes with M_q.

As we have shown that $\langle y^p \rangle$ permutes with each Sylow subgroup in $\mathfrak{S}$, it follows that y^p is $\mathfrak{S}$-quasinormal. By the choice of y we must have that

$y^p \in M$ and so lies in $\langle y \rangle \cap M = \{1\}$. It now follows that $|y| = |G : M| = p$ and the theorem follows.

In that a group is supersolvable if and only if each maximal subgroup has prime index [1], we have

COROLLARY. *The finite solvable group G is supersolvable if and only if for each maximal subgroup M of G, $G = MN_G^*(\mathfrak{S})$, where $\mathfrak{S}$ is some Sylow system of M.*

This may be restated as

COROLLARY. *The finite solvable group G is supersolvable if and only if every maximal subgroup M of G admits a cyclic supplement which permutes with each element in some Sylow system of M.*

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