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INEQUALITY BETWEEN THE BERGMAN METRIC AND CARATHÉODORY DIFFERENTIAL METRIC

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ABSTRACT. The author gives a short proof of an inequality between the Bergman metric and the Carathéodory differential metric on any complex manifold.

In this note we shall give a short proof of the following theorem.

THEOREM. *In any complex manifold M, the Bergman metric s_M is always greater than or equal to the Carathéodory differential metric c_M if M admits them.*

Let F be the set of square integrable holomorphic n-forms on a complex manifold M of dimension n. Then F is a separable Hilbert space with respect to the inner product $(\ ,\ )$. Let f be the element of F with $\|f\| = (f, f)^{1/2} \leqslant 1$ which maximizes $a_f = f \wedge \bar{f}$ at $z \in M$. Then $a_f = k_M(z, \bar{z})$, the Bergman kernel form of M. Let g be the element of F with $g(z) = 0$ and $\|g\| \leqslant 1$ which maximizes $b_g = \partial_\xi g \wedge \overline{\partial_\xi g}$ at z, where ∂_ξ denotes a holomorphic tangent vector in the ξ-direction. Then $b_g = k_M(z, \bar{z}) s_M^2(z, \xi)$. See [1] or [5]. Therefore, $s_M^2(z, \xi) = b_g / a_f$. If φ is a holomorphic function on M such that $\varphi(z) = 0$ and $|\varphi| \leqslant 1$, then φf is an element of F that can compete with g. Hence,

$$\begin{aligned} b_g &\geqslant \partial_\xi(\varphi f) \wedge \overline{\partial_\xi(\varphi f)} = (\partial_\xi \varphi) f \wedge \overline{(\partial_\xi \varphi) f} \\ &= |\partial_\xi \varphi|^2 a_f \quad \text{at } z \in M \end{aligned}$$

or

$$c_M^2(z, \xi) \leqslant s_M^2(z, \xi), \qquad z \in M.$$

This inequality has been previously proved in [3] and [4], and also in [2] using a different method. The proof given here is similar to that of [2].

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