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INTERVALS OF CONTINUA WHICH ARE HILBERT CUBES

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ABSTRACT. If P is a subcontinuum of a metric continuum X, then by the *interval of continua* $\mathcal{C}(P, X)$ we mean the space of all subcontinua of X which contain P (with the Hausdorff metric). We show that $\mathcal{C}(P, X)$ is often homeomorphic with the Hilbert cube.

In what follows X is a metric continuum and $\mathcal{C}(X)$ is the space of all subcontinua of X with the Hausdorff metric. If $P \in \mathcal{C}(X)$, then by the *interval of continua* $\mathcal{C}(P, X)$ we mean the subspace of $\mathcal{C}(X)$ consisting of all $A \in \mathcal{C}(X)$ which contain P. In [3], Curtis and Schori have shown that if X is locally connected and $X \setminus P$ is a nonvoid set containing no arc with interior, then the interval $\mathcal{C}(P, X)$ is homeomorphic with the Hilbert cube Q. In this note we will use a recent characterization of Q due to Torunczyk [8] to show that $\mathcal{C}(P, X)$ is often homeomorphic with Q without assuming X is locally connected.

In order to state Torunczyk's theorem we need his definition of a Z-map. A closed subset A of X is called a *Z-set in X* provided the identity map on X, 1_X, can be approximated by maps $f: X \to X$ such that $f(X) \cap A = \emptyset$. A map $f: X \to X$ is a *Z-map* if $f(X)$ is a Z-set in X. The following remarkable result is a special case of Theorem 1 in Torunczyk's paper.

1. THEOREM [TORUNCZYK]. *Suppose X is an AR (absolute retract) such that 1_X can be approximated by Z-maps. Then $X \approx Q$. ($\approx$ means is homeomorphic with.)*

This can be applied to intervals of continua because they are absolute retracts.

2. THEOREM. *$\mathcal{C}(P, X)$ is an AR.*

PROOF. It has been shown in [4] that $\mathcal{C}(P, X)$ is locally connected. Then using 4.3 in Kelley [6] and Lefschetz [7] it follows that $\mathcal{C}(P, X)$ is an ANR (absolute neighborhood retract). Now $\mathcal{C}(P, X)$ is contractible (to see this, let $f: [0, 1] \to \mathcal{C}(P, X)$ be any map with $f(1) = P$ and $f(0) = X$ and note that the homotopy $H: [0, 1] \times \mathcal{C}(P, X) \to \mathcal{C}(P, X)$ given by $H(t, A) = f(t) \cup A$ is a suitable contraction of $\mathcal{C}(P, X)$ to $\{X\}$), and so by Borsuk [1], $\mathcal{C}(P, X)$ is an AR.

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The following criterion provides a useful means of showing $\mathcal{C}(P, X) \approx Q$.

(*) *Suppose for each* $\varepsilon > 0$ *there is an* $A \in \mathcal{C}(P, X)$ *such that* A *is within* ε *of* P *in the Hausdorff metric and* $\mathcal{C}(A, X)$ *is a Z-set in* $\mathcal{C}(P, X)$. *Then* $\mathcal{C}(P, X) \approx Q$.

(*) follows from (1) and (2) upon noting that the map f: $\mathcal{C}(P, X) \to \mathcal{C}(A, X)$ given by $f(B) = A \cup B$ is a Z-map within ε of $1_{\mathcal{C}(P,X)}$.

We use the device of "passing to the quotient space" to simplify our discussions. Let X/P denote the quotient space obtained by identifying P to a point. Then the interval $\mathcal{C}(P, X)$ is homeomorphic in a natural way with the interval $\mathcal{C}(\{P\}, X/P)$. So to show $\mathcal{C}(P, X) \approx Q$ it will suffice to assume P is a singleton.

Call a pair (C, B) of points in $\mathcal{C}(\{p\}, X)$ an *annular pair* provided $C \subseteq B$ and $B \setminus C$ is a nonvoid open set in X.

3. Lemma. *If* $A \in \mathcal{C}(\{p\}, X)$ *and for each* $\delta > 0$, *there is an annular pair* (C, B) *with* $\operatorname{diam} B < \delta$ *and* $(B \setminus C) \cap A \neq \varnothing$, *then* $\mathcal{C}(A, X)$ *is a Z-set in* $\mathcal{C}(\{p\}, X)$.

Proof. Choose an annular pair (C, B) with $\operatorname{diam} B < \delta/2$ and $(B \setminus C) \cap A \neq \varnothing$, and consider the function g defined on $\mathcal{C}(B, X)$ by $g(D) = D \setminus (B \setminus C)$. Note that $g(D)$ is a closed subset of X since $B \setminus C$ is open in X. In fact, $g(D) \in \mathcal{C}(\{p\}, X)$. To see this, let $x \in D \setminus B$ and let K_x denote the closure of the component of x in $D \setminus B$. Since D is a continuum, we have that $K_x \cap B \neq \varnothing$. Also $K_x \subseteq g(D)$ since $g(D)$ is closed and contains $D \setminus B$. Hence $K_x \cap C \neq \varnothing$ and so $g(D) = C \cup (\bigcup_{x \in D \setminus B} K_x)$ is connected. Since $p \in D \cap C$ we have then that $g(D) \in \mathcal{C}(\{p\}, X)$ and since $(B \setminus C) \cap A \neq \varnothing$ we have that $g(D) \notin \mathcal{C}(A, X)$. It follows quickly from the definition of the Hausdorff metric that g is an isometry and that the Hausdorff distance from D to $g(D)$ is less than $\delta/2$.

To complete the argument let f: $\mathcal{C}(\{p\}, X) \to \mathcal{C}(B, X)$ be given by $f(D) = B \cup D$. Here again it follows from the definition of the Hausdorff metric that $d(D, f(D)) < \delta/2$ and $d(D, E) \geqslant d(f(D), f(E))$. Thus f is a continuous map within $\delta/2$ of $1_{\mathcal{C}(\{p\},X)}$. Now the composition h: $g \circ f$ is a map from $\mathcal{C}(\{p\}, X)$ to $\mathcal{C}(\{p\}, X)$ within δ of $1_{\mathcal{C}(\{p\},X)}$ for which $h(\mathcal{C}(\{p\}, X)) \cap \mathcal{C}(A, X) = \varnothing$. So $\mathcal{C}(A, X)$ is a Z-set in $\mathcal{C}(\{p\}, X)$.

The following theorem shows that, in certain settings, whether or not $\mathcal{C}(\{p\}, X) \approx Q$ depends entirely on the nature of the space X in a neighborhood of p.

4. Theorem. *Suppose* X *is locally connected at each point of an open set containing* p. *Then* $\mathcal{C}(\{p\}, X) \approx Q$ *if and only if* p *is not in the interior* (*relative to* X) *of a finite graph in* X.

Proof. If $p \in \operatorname{int} G$, where G is a finite graph in X, then one can construct in a natural way an open n-cell in $\mathcal{C}(\{p\}, X)$ about $\{p\}$, where n is the order of p in G. Conversely suppose p does not lie in the interior of any finite graph in X. Let $\varepsilon > 0$ be given. Choose a connected open set U about p whose diameter is less than $\varepsilon/2$. We can assume that X is locally connected at each point of the closure of U, which we will denote by A. To see that $\mathcal{C}(A, X)$ is a Z-set in $\mathcal{C}(\{p\}, X)$, let $\delta > 0$ be given. Choose a connected open set V about p such that $\operatorname{cl} V \subseteq U$ and $\operatorname{diam} V < \delta/2$. Using the local connectivity assumptions, cover the boundary of V with a finite number of continua $C_1, C_2, \ldots, C_n$ so that $p \notin C_i$ for each i and $\operatorname{diam}(\operatorname{cl} V \cup C_1 \cup \cdots \cup C_n) < \delta$. Let G be a finite graph in V which contains p and meets each C_i. G exists because V is arc-connected. Now let $B = \operatorname{cl} V \cup C_1 \cup \cdots \cup C_n$ and let $C = G \cup C_1 \cup C_2 \cup \cdots \cup C_n$. Note that B and C are members of $\mathcal{C}(\{p\}, X)$ and $B \supseteq C$ with $\operatorname{diam} B < \delta$. Also $B \setminus C$ is open in X since C contains the boundary of V and $(B \setminus C) \cap A$ is nonvoid since G does not contain p in its interior in X. Thus (C, B) is an annular pair which satisfies the hypothesis of (3). We conclude that $\mathcal{C}(A, X)$ is a Z-set in $\mathcal{C}(\{p\}, X)$.

We point out that the requirement that X be locally connected at each point in a neighborhood of p may be relaxed somewhat in (4). We illustrate with an example of a continuum X in the plane which is not locally connected and yet $\mathcal{C}(\{p\}, X) \approx Q$ for each $p \in X$.

Let T_0 be the convex hull of the points (0, 0), (0, 1) and $(-1, 1)$, and for each positive integer n, let T_n be the convex hull of the points (0, 0), $(1/2^{2n}, 1)$ and $(1/2^{2n+1}, 1)$. Let X be the union of all these closed triangular regions. Then X is a continuum which is locally connected at each point except along the edge E of T_0 from (0, 0) to (0, 1). It is easy to see that (4) gives $\mathcal{C}(\{p\}, X) \approx Q$ for any point p not in E. If $p \in E$ and $\varepsilon > 0$, let A_ε be the intersection of T_0 with the closed ε-disk about p and let B_ε be the boundary of A_ε. Then the pair $(B_\varepsilon, A_\varepsilon)$ is an annular pair as is easily verified. From (3) we conclude that $\mathcal{C}(A_\delta, X)$ is a Z-set in $\mathcal{C}(\{p\}, X)$. Then by (*), $\mathcal{C}(\{p\}, X) \approx Q$.

The result in (4) can be restated: *Suppose X is locally connected in a neighborhood of p. Then $\mathcal{C}(\{p\}, X) \approx Q$ if and only if $\mathcal{C}(\{p\}, X)$ is infinite dimensional at $\{p\}$.*

In the remainder we give another method for showing that $\mathcal{C}(\{p\}, X) \approx Q$ which will, for example, show that if X is the *Cantor fan* (cone over the Cantor set), then $\mathcal{C}(\{p\}, X) \approx Q$ where p is the vertex of the cone. The criterion which applies here is an immediate consequence of (1) and (2).

(**) Suppose for each $\varepsilon > 0$, there is an $A \in \mathcal{C}(\{p\}, X)$ and a map f: $\mathcal{C}(\{p\}, X) \to \mathcal{C}(\{p\}, A)$ such that $\mathcal{C}(\{p\}, A)$ is a Z-set in $\mathcal{C}(\{p\}, X)$ and f is within ε of the identity on $\mathcal{C}(\{p\}, X)$. Then $\mathcal{C}(\{p\}, X) \approx Q$.

If $A \in \mathcal{C}(\{p\}, X)$, then A is said to *cover* p provided the interior of $\mathcal{C}(\{p\}, A)$ in $\mathcal{C}(\{p\}, X)$ contains $\{p\}$.

5. Lemma. *Suppose $A \in \mathcal{C}(\{p\}, X)$ does not cover p. Then $\mathcal{C}(\{p\}, A)$ is a Z-set in $\mathcal{C}(\{p\}, X)$.*

Proof. Let $\varepsilon > 0$. Since A does not cover p there is a $B \in \mathcal{C}(\{p\}, X)$ such that $\operatorname{diam} B < \varepsilon$ and $B \cap (X \setminus A) \neq \varnothing$. Then the map f: $\mathcal{C}(\{p\}, X) \to \mathcal{C}(B, X)$ given by $f(D) = B \cup D$ is ε-close to the identity on $\mathcal{C}(\{p\}, X)$ since diam $B < \varepsilon$. The condition $B \cap X \setminus A \neq \varnothing$ implies the image of f misses $\mathcal{C}(\{p\}, A)$. So $\mathcal{C}(\{p\}, A)$ is a Z-set in $\mathcal{C}(\{p\}, X)$.

6. **Theorem.** *Suppose the identity on X can be approximated by maps f such that $f(p) = p$ and $f(X)$ does not cover p. Then $\mathcal{C}(\{p\}, X) \approx Q$.*

Proof. Since $f(p) = p$, the induced map $\bar{f}$ on $\mathcal{C}(X)$ given by $\bar{f}(A) = \{f(x) | x \in A\}$ takes $\mathcal{C}(\{p\}, X)$ into $\mathcal{C}(\{p\}, f(X))$. Clearly if $d(f, 1_X) < \varepsilon$, then $d(\bar{f}, 1_{\mathcal{C}(\{p\}, X)}) < \varepsilon$. Further if $f(X)$ does not cover p, then the image of $\bar{f}$, being a closed subset of the Z-set $\mathcal{C}(\{p\}, f(X))$, is a Z-set in $\mathcal{C}(\{p\}, X)$. Now apply (**).

This theorem has numerous applications.

7. **Corollary.** *Suppose X is a compact metric space (X not assumed connected here) such that the identity map on X can be approximated by maps f: $X \to X$ such that $f(X) \neq X$. Then $\mathcal{C}(\{p\}, \mathrm{Cone}\ X) \approx Q$, where p is the vertex of* Cone X, *the cone over X.*

Proof. Embed X in Q and realize Cone X as the subspace $\{(t, tx) | t \in [0, 1], x \in X\}$ of $[0, 1] \times Q$. Now any map f: $X \to X$ induces a map f': Cone $X \to$ Cone X given by $f'(t, tx) = (t, tf(x))$. Note that $f'(p) = p$ where $p = (0, 0, \ldots)$ is the vertex of Cone X and that $d(f', 1_{\mathrm{Cone}\ X}) = d(f, 1_X)$. Also if $f(X) \neq X$, then $f'(\mathrm{Cone}\ X)$ does not cover p. Hence (6) applies.

Thus as mentioned above, (7) can be used to show that $C(\{p\}, X) \approx Q$ when X is the Cantor fan and p is the vertex. In this case we can also use (8) below.

We can often use (6) to show that $\mathcal{C}(p, X) \approx Q$ when X is a *dendroid*; that is, a hereditarily unicoherent arc-connected continuum. For example, let X be the continuum in the plane consisting of the union of all segments from (0, 0) to $(1, t)$ or from (0, 1) to $(-t, 0)$ where t runs over the Cantor ternary set C. To see that $\mathcal{C}(\{(0, 0)\}, X) \approx Q$, let f_n: $X \to X$ be defined by $f_n((r, r \cdot t)) = (r, 0)$ if $r \in (0, 1]$ and $t \in C \cap [0, 1/3^n]$ and let $f_n(x) = x$ otherwise. Then f_n is a retraction which moves no point more than $1/3^n$. Further $f_n(X)$ does not cover (0, 0), since the segment A_ε from $(\varepsilon, \varepsilon/3^n)$ to (0, 0) meets $X \setminus f_n(X)$ for all $\varepsilon > 0$. Since the maps f_n approximate 1_X, (6) applies.

If X is a dendroid *smooth at p* (see [2] for the definition of smooth dendroid) we have an analogue of (4).

8. **Theorem.** *Suppose X is a dendroid smooth at p. Then $\mathcal{C}(\{p\}, X) \approx Q$ if and only if p is not in the interior of a finite tree in X.*

Proof. J. B. Fugate has shown that for each $\varepsilon > 0$, there is an ε-retraction from X onto a finite tree T in X [5]. In his construction, $p \in T$. The image of such an ε-retraction cannot cover p. This is because X is locally connected at p and, by assumption, $p \notin \mathrm{int}\ T$. Hence $\mathcal{C}(\{p\}, X) \approx Q$.

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