

UNIFORM ASYMPTOTIC STABILITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS

T. A. BURTON

ABSTRACT. The classical uniform asymptotic stability result for a system of functional differential equations

$$(1) \quad x' = F(t, x_t)$$

calls for a Liapunov functional $V(t, \phi)$ satisfying $W(|\phi(0)|) < V(t, \phi) < W_1(|\phi(0)|) + W_2(\|\phi\|)$, $V'(t) < -W_3(|\phi(0)|)$, and $|f(t, x_t)|$ bounded for $\|x_t\|$ bounded. We show that it is not necessary to require $|f(t, x_t)|$ bounded. Here, $\|\cdot\|$ is the L^2 -norm.

1. Introduction and notation. We consider a system of functional differential equations

$$(1) \quad x'(t) = F(t, x_t)$$

and obtain conditions on a Liapunov functional to insure that the zero solution is uniformly asymptotically stable. This work generalizes the classical result in that we do not require the usual bound on F for x_t bounded. For reference, our discussion here follows closely that of Yoshizawa [1, pp. 183–192]. Our result is a generalization of his theorem on p. 192.

For $x \in R^n$, $|x|$ will be the $\max|x_i|$. For a given $h > 0$, C denotes the space of continuous functions mapping $[-h, 0]$ into R^n and for $\phi \in C$, $\|\phi\| = \sup_{-h < \theta \leq 0} |\phi(\theta)|$. C_H denotes the set of $\phi \in C$ with $\|\phi\| \leq H$. If x is a continuous function of u defined on $-h \leq u < A$, $A > 0$, and if t is a fixed number satisfying $0 \leq t < A$, then x_t denotes the restriction of x to the interval $[t - h, t]$ so that x_t is an element of C defined by $x_t(\theta) = x(t + \theta)$ for $-h \leq \theta \leq 0$.

In (1), $x'(t)$ denotes the right-hand derivative of x at t and $F(t, \phi) \in R^n$ is defined on $[0, \infty) \times C_H$.

We denote by $x(t_0, \phi)$ a solution of (1) with initial condition $\phi \in C_H$ where $x_{t_0}(t_0, \phi) = \phi$ and we denote by $x(t; t_0, \phi)$ the value of $x(t_0, \phi)$ at t .

It is assumed that $F(t, \phi)$ is continuous on $[0, \infty) \times C_H$ so that a solution will exist for each continuous initial condition. For continuation of solutions, we suppose that F takes closed bounded sets of $[0, \infty) \times C_H$ into closed bounded sets of R^n .

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Let $V(t, \phi)$ be a continuous functional defined for $t \geq 0, \phi \in C_H$. The upper right-hand derivative of V along solutions of (1) is defined to be

$$V'_{(1)}(t, x_t(t_0, \phi)) = \overline{\lim}_{\delta \rightarrow 0^+} \{V(t + \delta, x_{t+\delta}(t_0, \phi)) - V(t, x_t(t_0, \phi))\} / \delta.$$

If V satisfies a Lipschitz condition in the second argument, then this limit is uniquely determined.

We suppose that $F(t, 0) = 0$, and we say that the zero solution is uniformly stable if for each $\epsilon > 0$ there exists $\delta > 0$ so that $t_0 \geq 0, t \geq t_0$, and $\|\phi\| < \delta$ imply $|x(t; t_0, \phi)| < \epsilon$. If, in addition, there exists $\delta > 0$ such that for any $\epsilon > 0$ there exists $T > 0$ such that $t_0 \geq 0, t \geq t_0 + T$, and $\|\phi\| < \delta$ imply $|x(t; t_0, \phi)| < \epsilon$, then $x = 0$ is uniformly asymptotically stable (UAS).

If $\phi \in C_H$, then

$$\|\phi\| = \left[\sum_{i=1}^n \int_{-h}^0 \phi_i^2(s) ds \right]^{1/2}$$

where ϕ_i is the i th component of ϕ . Finally, W, W_1, W_2 , and W_3 will denote continuous functions from $[0, \infty)$ into $[0, \infty)$ with $W_i(0) = 0, W_i(r) > 0$ if $r > 0$, and W_i nondecreasing.

2. Uniform asymptotic stability. The following lemma is used in the proof of our result.

LEMMA. *Let $\{x_n\}$ be a sequence of continuous functions having continuous derivatives, $x_n: [0, 1] \rightarrow [0, 1]$. Let $g: [0, \infty) \rightarrow [0, \infty)$ be continuous, $g(0) = 0, g(r) > 0$ if $r > 0$, and let g be nondecreasing. If there exists $\alpha > 0$ with $\int_0^1 x_n(t) dt \geq \alpha$ for all n , then there exists $\beta > 0$ with $\int_0^1 g(x_n(t)) dt \geq \beta$ for all n .*

PROOF. Let $M_k = \{t | x_k(t) \geq \alpha/2\}$, M_k^c be the complement of M_k on $[0, 1]$, and let m_k be the measure of M_k . Then $m_k \geq \alpha/2$. To see this, if $m_k < \alpha/2$, then

$$\alpha \leq \int_0^1 x_k(t) dt = \int_{M_k} x_k(t) dt + \int_{M_k^c} x_k(t) dt < \frac{\alpha}{2} + \frac{\alpha}{2},$$

a contradiction. (The first $\alpha/2$ follows from $m_k < \alpha/2$ and $x_k(t) \leq 1$. The second $\alpha/2$ follows from $\text{meas } M_k^c \leq 1$ and $x_k(t) \leq \alpha/2$.) Hence, for any k , we have

$$\begin{aligned} \int_0^1 g(x_k(t)) dt &\geq \int_{M_k} g(x_k(t)) dt \geq \int_{M_k} g(\alpha/2) dt \\ &= m_k g(\alpha/2) \geq g(\alpha/2) \alpha/2 \stackrel{\text{def}}{=} \beta. \end{aligned}$$

This completes the proof.

The following result is known when $|F(t, \phi)| \leq L$ for $0 \leq t < \infty$ and $\|\phi\| < H$.

THEOREM 1. *Suppose there is a continuous functional $V(t, \phi)$, locally*

Lipschitz in ϕ , defined on $0 \leq t < \infty, \|\phi\| < H$ which satisfies:

- (i) $W(|\phi(0)|) \leq V(t, \phi) \leq W_1(|\phi(0)|) + W_2(\|\phi\|)$ and
- (ii) $V'_{(1)}(t, \phi) \leq -W_3(|\phi(0)|)$.

Then the zero solution of (1) is uniformly asymptotically stable.

PROOF. For a given $\epsilon > 0$ ($\epsilon < H$), choose $\delta > 0$ so that $W_1(\delta) + W_2([\delta^2nh]^{1/2}) < W(\epsilon)$. Then for an initial function $\phi \in C_H$ satisfying $\|\phi\| < \delta$ we have $V'_{(1)}(t, x_t(t_0, \phi)) \leq 0$ and so if $x(t) = x(t; t_0, \phi)$, then

$$\begin{aligned} W(|x(t)|) &\leq V(t, x_t(t_0, \phi)) \leq V(t_0, \phi) \\ &\leq W_1(|\phi(0)|) + W_2(\|\phi\|) \\ &< W_1(\delta) + W_2([\delta^2nh]^{1/2}) < W(\epsilon) \end{aligned}$$

so that $|x(t)| < \epsilon$ for $t \geq t_0$ by the monotonicity of W . Thus, $x = 0$ is uniformly stable.

Next, given $H > 0$, find the δ of uniform stability for $\min[H, 1]$. To complete the proof of UAS we must show that for each $\epsilon > 0$ there exists $T > 0$ such that $t_0 \geq 0, t > t_0 + T$, and $\|\phi\| < \delta$ imply $|x(t; t_0, \phi)| < \epsilon$.

To this end, let $\epsilon > 0$ be given and choose $\epsilon_1 > 0$ with $\epsilon_1 < \epsilon$ and $W_1(\epsilon_1) < W(\epsilon)/2$. Now there exists $\epsilon_2 > 0$ such that $\|x_t(t_0, \phi)\| < \epsilon_2$ implies $W_2(\|x_t(t_0, \phi)\|) < W(\epsilon)/2$. That is,

$$\begin{aligned} W_2(\|x_t(t_0, \phi)\|) &= W_2\left[\left(\int_{-h}^0 \sum_{i=1}^n x_i^2(t+s)ds\right)^{1/2}\right] \\ &\leq W_2([nh\epsilon_2^2]^{1/2}) < W(\epsilon)/2 \end{aligned}$$

if $\|x_t(t_0, \phi)\| < \epsilon_2$ and ϵ_2 is small enough. Let $\epsilon_3 = \min[\epsilon_1, \epsilon_2]$.

With these choices, notice that if a solution satisfies $|x(t; t_0, \phi)| < \epsilon_3$ on an interval of length h , we then have $|x(t; t_0, \phi)| < \epsilon$ for all future time. This is merely the uniform stability argument once more. For, if $|x(t; t_0, \phi)| < \epsilon_3$ for $t_1 - h \leq t \leq t_1$, then this is equivalent to $\|x_t(t_0, \phi)\| < \epsilon_3 \leq \epsilon_2$. Then for $t \geq t_1$ we have

$$\begin{aligned} W(|x(t; t_0, \phi)|) &\leq V(t, x_t(t_0, \phi)) \leq V(t_1, x_t(t_0, \phi)) \\ &\leq W_1(|x(t_1; t_0, \phi)|) + W_2(\|x_t(t_0, \phi)\|) \\ &< W_1(\epsilon_1) + W(\epsilon)/2 < W(\epsilon). \end{aligned}$$

Now recall that $\phi \in C_H, \|\phi\| < \delta$, and $t \geq t_0$ yields $V(t, x_t(t_0, \phi)) \leq V(t_0, \phi) \leq W_1(\delta) + W_2([\delta^2nh]^{1/2}) = \text{def } \mu$. Also, $V'(t, x_t(t_0, \phi)) \leq -W_3(|x(t, t_0, \phi)|)$ so there exists $T_1 > h$ such that $|x(t; t_0, \phi)| \geq \epsilon_3$ must fail for some value of t in each interval of length T_1 . To see this, note that if $|x(t, t_0, \phi)| \geq \epsilon_3$ on an interval $[t_1, t_2]$, then

$$\begin{aligned} V(t_2, x_{t_2}(t_0, \phi)) &\leq V(t_1, x_{t_1}(t_0, \phi)) - W_3(\epsilon_3)(t_2 - t_1) \\ &\leq \mu - W_3(\epsilon_3)(t_2 - t_1) < 0 \end{aligned}$$

if $t_2 - t_1 > \mu/W_3(\epsilon_3)$, a contradiction to $V \geq 0$.

Thus, for each $\phi \in C_H$ with $\|\phi\| < \delta$ there is a sequence $\{t_n\}$ monotone to $+\infty$ with $|x(t_n; t_0, \phi)| < \epsilon_3$. By choosing a subsequence we can say $t_n + h < t_{n+1}$. In particular, when ϕ and t_0 are given with $\|\phi\| < \delta$ and $t_0 \geq 0$, we may choose $t_1 \in [t_0, t_0 + T_1]$, $t_2 \in [t_0 + 2T_1, t_0 + 3T_1]$, $t_3 \in [t_0 + 4T_1, t_0 + 5T_1]$, The t_i depend on ϕ , but the indicated interval in which t_i lies is independent of ϕ whenever $\|\phi\| < \delta$.

Consider now the sequence of functions $\{x_k(t_0, \phi)\}$ and examine those members satisfying $W_2(\|x_k(t_0, \phi)\|) \geq W(\epsilon)/2$. This yields $\|x_k(t_0, \phi)\| \geq \sqrt{\bar{\alpha}}$, for some $\bar{\alpha} > 0$ so that $\sum_{i=1}^n \int_{t_k-h}^0 x_i^2(t_k + s) ds \geq \bar{\alpha}$. From this we obtain

$$(*) \quad \sum_{i=1}^n (1/n) \int_{t_k-h}^{t_k} x_i^2(s) ds \geq \bar{\alpha}/n \stackrel{\text{def}}{=} \alpha$$

where the x_i are the components of $x_k(t_0, \phi)$.

We now play the function W_3 against W_2 . For $t > t_k$ we have

$$\begin{aligned} V(t, x_t(t_0, \phi)) &\leq \mu - \int_{t_0}^t W_3(|x(u; t_0, \phi)|) du \\ &\leq \mu - \sum_{i=2}^k \int_{t_i-h}^{t_i} W_3(|x(t; t_0, \phi)|) dt \\ &\leq \mu - \sum_{i=2}^k \int_{t_i-h}^{t_i} W_3\left([1/n] \sum_{j=1}^n x_j^2(t; t_0, \phi)\right) dt \end{aligned}$$

as $x_j^2(t; t_0, \phi) \leq H^2 < 1$.

But from (*) and the lemma we conclude that

$$\int_{t_i-h}^{t_i} W_3\left([1/n] \sum_{j=1}^n x_j^2(t; t_0, \phi)\right) dt > \beta$$

for some $\beta > 0$ whenever (*) holds. Thus, if (*) holds for x_{t_1}, \dots, x_{t_k} , then $V(t, x_t(t_0, \phi)) \leq \mu - (k - 1)\beta$. Now there exists N such that $k > N$ yields $\mu - (k - 1)\beta < 0$.

As $V \geq 0$, we conclude that (*) fails for some $x_{t_i}(t_0, \phi)$ for $1 \leq i \leq N$. Thus, for that i , we have $W_2(\|x_{t_i}(t_0, \phi)\|) < W(\epsilon)/2$ and, consequently, as $|x(t_i; t_0, \phi)| < \epsilon_3$ we have $|x(t; t_0, \phi)| < \epsilon$ for $t \geq t_i$.

Finally, we claim that $t_N < t_0 + 2NT_1$ so that $T = 2NT_1$ suffices. To see this, recall that $|x(t; t_0, \phi)| \geq \epsilon_3$ fails for some t in every interval of length T_1 and $T_1 > h$. This completes the proof.

As a simple example, let

$$x'(t) = -(t + 3)x(t) + x(t - 1)$$

and

$$V(t, x_t) = x^2(t)/2 + \int_{t-1}^t x^2(s) ds.$$

Then

$$\begin{aligned}V'(t, x_t) &= -(t + 3)x^2(t) + x(t)x(t - 1) + x^2(t) - x^2(t - 1) \\ &< -x^2(t) = -W_3(|x(t)|)\end{aligned}$$

when $W_3(r) = r^2$. The upper and lower wedges on V are clear. We conclude that the zero solution is UAS.

REFERENCES

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DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY, CARBONDALE, ILLINOIS 62901