PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 68, Number 2, February 1978

UNIFORM ASYMPTOTIC STABILITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS

T. A. BURTON

ABSTRACT. The classical uniform asymptotic stability result for a system of functional differential equations

(1) $$x' = F(t, x_t)$$

calls for a Liapunov functional $V(t, \phi)$ satisfying $W(|\phi(0)|) < V(t, \phi) < W_1(|\phi(0)|) + W_2(|||\phi|||)$, $V'_{(1)} < -W_3(|\phi(0)|)$, and $|f(t, x_t)|$ bounded for $|||x_t|||$ bounded. We show that it is not necessary to require $|f(t, x_t)|$ bounded. Here, $|||\cdot|||$ is the L^2-norm.

1. Introduction and notation. We consider a system of functional differential equations

(1) $$x'(t) = F(t, x_t)$$

and obtain conditions on a Liapunov functional to insure that the zero solution is uniformly asymptotically stable. This work generalizes the classical result in that we do not require the usual bound on F for x_t bounded. For reference, our discussion here follows closely that of Yoshizawa [**1**, pp. 183–192]. Our result is a generalization of his theorem on p. 192.

For $x \in R^n$, $|x|$ will be the $\max|x_i|$. For a given $h > 0$, C denotes the space of continuous functions mapping $[-h, 0]$ into R^n and for $\phi \in C$, $\|\phi\| = \sup_{-h \leq \theta \leq 0}|\phi(\theta)|$. C_H denotes the set of $\phi \in C$ with $\|\phi\| \leq H$. If x is a continuous function of u defined on $-h \leq u < A$, $A > 0$, and if t is a fixed number satisfying $0 \leq t < A$, then x_t denotes the restriction of x to the interval $[t - h, t]$ so that x_t is an element of C defined by $x_t(\theta) = x(t + \theta)$ for $-h \leq \theta \leq 0$.

In (1), $x'(t)$ denotes the right-hand derivative of x at t and $F(t, \phi) \in R^n$ is defined on $[0, \infty) \times C_H$.

We denote by $x(t_0, \phi)$ a solution of (1) with initial condition $\phi \in C_H$ where $x_{t_0}(t_0, \phi) = \phi$ and we denote by $x(t; t_0, \phi)$ the value of $x(t_0, \phi)$ at t.

It is assumed that $F(t, \phi)$ is continuous on $[0, \infty) \times C_H$ so that a solution will exist for each continuous initial condition. For continuation of solutions, we suppose that F takes closed bounded sets of $[0, \infty) \times C_H$ into closed bounded sets of R^n.

Received by the editors June 30, 1977.

AMS (MOS) subject classifications (1970). Primary 34K20.

Key words and phrases. Functional differential equations, Liapunov functionals, uniform asymptotic stability.

Let $V(t, \phi)$ be a continuous functional defined for $t \geqslant 0$, $\phi \in C_H$. The upper right-hand derivative of V along solutions of (1) is defined to be

$$V'_{(1)}(t, x_t(t_0, \phi)) = \overline{\lim_{\delta \to 0^+}} \{V(t+\delta, x_{t+\delta}(t_0, \phi)) - V(t, x_t(t_0, \phi))\}/\delta.$$

If V satisfies a Lipschitz condition in the second argument, then this limit is uniquely determined.

We suppose that $F(t, 0) = 0$, and we say that the zero solution is uniformly stable if for each $\varepsilon > 0$ there exists $\delta > 0$ so that $t_0 \geqslant 0$, $t \geqslant t_0$, and $\|\phi\| < \delta$ imply $|x(t; t_0, \phi)| < \varepsilon$. If, in addition, there exists $\delta > 0$ such that for any $\varepsilon > 0$ there exists $T > 0$ such that $t_0 \geqslant 0$, $t \geqslant t_0 + T$, and $\|\phi\| < \delta$ imply $|x(t; t_0, \phi)| < \varepsilon$, then $x = 0$ is uniformly asymptotically stable (UAS).

If $\phi \in C_H$, then

$$|||\phi||| = \left[\sum_{i=1}^{n} \int_{-h}^{0} \phi_i^2(s)\, ds\right]^{1/2}$$

where ϕ_i is the ith component of ϕ. Finally, W, W_1, W_2, and W_3 will denote continuous functions from $[0, \infty)$ into $[0, \infty)$ with $W_i(0) = 0$, $W_i(r) > 0$ if $r > 0$, and W_i nondecreasing.

2. Uniform asymptotic stability. The following lemma is used in the proof of our result.

LEMMA. *Let $\{x_n\}$ be a sequence of continuous functions having continuous derivatives, x_n: $[0, 1] \to [0, 1]$. Let g: $[0, \infty) \to [0, \infty)$ be continuous, $g(0) = 0$, $g(r) > 0$ if $r > 0$, and let g be nondecreasing. If there exists $\alpha > 0$ with $\int_0^1 x_n(t)\, dt \geqslant \alpha$ for all n, then there exists $\beta > 0$ with $\int_0^1 g(x_n(t))\, dt \geqslant \beta$ for all n.*

PROOF. Let $M_k = \{t | x_k(t) \geqslant \alpha/2\}$, M_k^c be the complement of M_k on [0, 1], and let m_k be the measure of M_k. Then $m_k \geqslant \alpha/2$. To see this, if $m_k < \alpha/2$, then

$$\alpha \leqslant \int_0^1 x_k(t)\, dt = \int_{M_k} x_k(t)\, dt + \int_{M_k^c} x_k(t)\, dt < \frac{\alpha}{2} + \frac{\alpha}{2},$$

a contradiction. (The first $\alpha/2$ follows from $m_k < \alpha/2$ and $x_k(t) \leqslant 1$. The second $\alpha/2$ follows from meas $M_k^c \leqslant 1$ and $x_k(t) \leqslant \alpha/2$.) Hence, for any k, we have

$$\int_0^1 g(x_k(t))\, dt \geqslant \int_{M_k} g(x_k(t))\, dt \geqslant \int_{M_k} g(\alpha/2)\, dt$$
$$= m_k g(\alpha/2) \geqslant g(\alpha/2)\alpha/2 \overset{\text{def}}{=} \beta.$$

This completes the proof.

The following result is known when $|F(t, \phi)| \leqslant L$ for $0 \leqslant t < \infty$ and $\|\phi\| < H$.

THEOREM 1. *Suppose there is a continuous functional $V(t, \phi)$, locally*

Lipschitz in ϕ, defined on $0 \leqslant t < \infty$, $\|\phi\| < H$ *which satisfies*:

(i) $W(|\phi(0)|) \leqslant V(t, \phi) \leqslant W_1(|\phi(0)|) + W_2(|||\phi|||)$ *and*

(ii) $V'_{(1)}(t, \phi) \leqslant -W_3(|\phi(0)|)$.

Then the zero solution of (1) *is uniformly asymptotically stable.*

PROOF. For a given $\varepsilon > 0$ ($\varepsilon < H$), choose $\delta > 0$ so that $W_1(\delta) + W_2([\delta^2 nh]^{1/2}) < W(\varepsilon)$. Then for an initial function $\phi \in C_H$ satisfying $\|\phi\| < \delta$ we have $V'_{(1)}(t, x_t(t_0, \phi)) \leqslant 0$ and so if $x(t) = x(t; t_0, \phi)$, then

$$\begin{aligned} W\big(|x(t)|\big) &\leqslant V(t, x_t(t_0, \phi)) \leqslant V(t_0, \phi) \\ &\leqslant W_1\big(|\phi(0)|\big) + W_2\big(|||\phi|||\big) \\ &< W_1(\delta) + W_2\big([\delta^2 nh]^{1/2}\big) < W(\varepsilon) \end{aligned}$$

so that $|x(t)| < \varepsilon$ for $t \geqslant t_0$ by the monotonicity of W. Thus, $x = 0$ is uniformly stable.

Next, given $H > 0$, find the δ of uniform stability for $\min[H, 1]$. To complete the proof of UAS we must show that for each $\varepsilon > 0$ there exists $T > 0$ such that $t_0 \geqslant 0$, $t > t_0 + T$, and $\|\phi\| < \delta$ imply $|x(t; t_0, \phi)| < \varepsilon$.

To this end, let $\varepsilon > 0$ be given and choose $\varepsilon_1 > 0$ with $\varepsilon_1 < \varepsilon$ and $W_1(\varepsilon_1) < W(\varepsilon)/2$. Now there exists $\varepsilon_2 > 0$ such that $\|x_t(t_0, \phi)\| < \varepsilon_2$ implies $W_2(|||x_t(t_0, \phi)|||) < W(\varepsilon)/2$. That is,

$$\begin{aligned} W_2\big(|||x_t(t_0, \phi)|||\big) &= W_2\left(\left[\int_{-h}^{0} \sum_{i=1}^{n} x_i^2(t+s)\,ds\right]^{1/2}\right) \\ &\leqslant W_2\big([nh\varepsilon_2^2]^{1/2}\big) < W(\varepsilon)/2 \end{aligned}$$

if $\|x_t(t_0, \phi)\| < \varepsilon_2$ and ε_2 is small enough. Let $\varepsilon_3 = \min[\varepsilon_1, \varepsilon_2]$.

With these choices, notice that if a solution satisfies $|x(t; t_0, \phi)| < \varepsilon_3$ on an interval of length h, we then have $|x(t; t_0, \phi)| < \varepsilon$ for all future time. This is merely the uniform stability argument once more. For, if $|x(t; t_0, \phi)| < \varepsilon_3$ for $t_1 - h \leqslant t \leqslant t_1$, then this is equivalent to $\|x_{t_1}(t_0, \phi)\| < \varepsilon_3 \leqslant \varepsilon_2$. Then for $t \geqslant t_1$ we have

$$\begin{aligned} W\big(|x(t; t_0, \phi)|\big) &\leqslant V(t, x_t(t_0, \phi)) \leqslant V\big(t_1, x_{t_1}(t_0, \phi)\big) \\ &\leqslant W_1\big(|x(t_1; t_0, \phi)|\big) + W_2\big(|||x_{t_1}(t_0, \phi)|||\big) \\ &< W_1(\varepsilon_1) + W(\varepsilon)/2 < W(\varepsilon). \end{aligned}$$

Now recall that $\phi \in C_H$, $\|\phi\| < \delta$, and $t \geqslant t_0$ yields $V(t, x_t(t_0, \phi)) \leqslant V(t_0, \phi) \leqslant W_1(\delta) + W_2([nh\delta^2]^{1/2}) \overset{\text{def}}{=} \mu$. Also, $V'(t, x_t(t_0, \phi)) \leqslant -W_3(|x(t, t_0, \phi)|)$ so there exists $T_1 > h$ such that $|x(t; t_0, \phi)| \geqslant \varepsilon_3$ must fail for some value of t in each interval of length T_1. To see this, note that if $|x(t, t_0, \phi)| \geqslant \varepsilon_3$ on an interval $[t_1, t_2]$, then

$$\begin{aligned} V\big(t_2, x_{t_2}(t_0, \phi)\big) &\leqslant V\big(t_1, x_{t_1}(t_0, \phi)\big) - W_3(\varepsilon_3)(t_2 - t_1) \\ &\leqslant \mu - W_3(\varepsilon_3)(t_2 - t_1) < 0 \end{aligned}$$

if $t_2 - t_1 > \mu/W_3(\varepsilon_3)$, a contradiction to $V \geqslant 0$.

Thus, for each $\phi \in C_H$ with $\|\phi\| < \delta$ there is a sequence $\{t_n\}$ monotone to $+\infty$ with $|x(t_n; t_0, \phi)| < \varepsilon_3$. By choosing a subsequence we can say $t_n + h < t_{n+1}$. In particular, when ϕ and t_0 are given with $\|\phi\| < \delta$ and $t_0 \geqslant 0$, we may choose $t_1 \in [t_0, t_0 + T_1]$, $t_2 \in [t_0 + 2T_1, t_0 + 3T_1]$, $t_3 \in [t_0 + 4T_1, t_0 + 5T_1], \ldots$. The t_i depend on ϕ, but the indicated interval in which t_i lies is independent of ϕ whenever $\|\phi\| < \delta$.

Consider now the sequence of functions $\{x_{t_k}(t_0, \phi)\}$ and examine those members satisfying $W_2(|||x_{t_k}(t_0, \phi)|||) \geqslant W(\varepsilon)/2$. This yields $|||x_{t_k}(t_0, \phi)||| \geqslant \sqrt{\bar{\alpha}}$, for some $\bar{\alpha} > 0$ so that $\sum_{i=1}^{n} \int_{-h}^{0} x_i^2(t_k + s)\, ds \geqslant \bar{\alpha}$. From this we obtain

$$(*) \qquad \sum_{i=1}^{n} (1/n) \int_{t_k - h}^{t_k} x_i^2(s)\, ds \geqslant \bar{\alpha}/n \overset{\text{def}}{=} \alpha$$

where the x_i are the components of $x_{t_k}(t_0, \phi)$.

We now play the function W_3 against W_2. For $t > t_k$ we have

$$\begin{aligned} V(t, x_t(t_0, \phi)) &\leqslant \mu - \int_{t_0}^{t} W_3\big(|x(u; t_0, \phi)|\big)\, du \\ &\leqslant \mu - \sum_{i=2}^{k} \int_{t_i - h}^{t_i} W_3\big(|x(t; t_0, \phi)|\big)\, dt \\ &\leqslant \mu - \sum_{i=2}^{k} \int_{t_i - h}^{t_i} W_3\left([1/n] \sum_{j=1}^{n} x_j^2(t; t_0, \phi)\right) dt \end{aligned}$$

as $x_j^2(t; t_0, \phi) \leqslant H^2 < 1$.

But from $(*)$ and the lemma we conclude that

$$\int_{t_i - h}^{t_i} W_3\left([1/n] \sum_{j=1}^{n} x_j^2(t; t_0, \phi)\right) \geqslant \beta$$

for some $\beta > 0$ whenever $(*)$ holds. Thus, if $(*)$ holds for $x_{t_1}, \ldots, x_{t_k}$, then $V(t, x_t(t_0, \phi)) \leqslant \mu - (k-1)\beta$. Now there exists N such that $k > N$ yields $\mu - (k-1)\beta < 0$.

As $V \geqslant 0$, we conclude that $(*)$ fails for some $x_{t_i}(t_0, \phi)$ for $1 \leqslant i \leqslant N$. Thus, for that i, we have $W_2(|||x_{t_i}(t_0, \phi)|||) < W(\varepsilon)/2$ and, consequently, as $|x(t_i; t_0, \phi)| < \varepsilon_3$ we have $|x(t; t_0, \phi)| < \varepsilon$ for $t \geqslant t_i$.

Finally, we claim that $t_N < t_0 + 2NT_1$ so that $T = 2NT_1$ suffices. To see this, recall that $|x(t; t_0, \phi)| \geqslant \varepsilon_3$ fails for some t in every interval of length T_1 and $T_1 > h$. This completes the proof.

As a simple example, let

$$x'(t) = -(t+3)x(t) + x(t-1)$$

and

$$V(t, x_t) = x^2(t)/2 + \int_{t-1}^{t} x^2(s)\, ds.$$

Then

$$\begin{aligned} V'(t, x_t) &= -(t+3)x^2(t) + x(t)x(t-1) + x^2(t) - x^2(t-1) \\ &\leq -x^2(t) = -W_3(|x(t)|) \end{aligned}$$

when $W_3(r) = r^2$. The upper and lower wedges on V are clear. We conclude that the zero solution is UAS.

References

1. T. Yoshizawa, *Stability theory by Lyapunov's second method*, Math. Soc. Japan, 1966.

DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY, CARBONDALE, ILLINOIS 62901