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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO ABSTRACT VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. The existence and uniqueness of solutions to the equation $u(t) + \int_0^t a(t-s)Au(s)\,ds \ni f(t)$, where A is a maximal monotone operator, is proved under various restrictions on A and f.

I. Introduction. This paper discusses the existence and uniqueness of solutions to the abstract Volterra integral equation

$$u(t) + \int_0^t a(t-s)Au(s)\,ds \ni f(t), \tag{1.1}$$

where A is a possibly multiple valued maximal monotone operator from the real Hilbert space H into H, f maps the interval $[0, T]$ into H, and $a(t)$ is a real valued differentiable function defined on $[0, T]$, such that $a(0) > 0$. Equations similar to (1.1) arise in the study of heat transfer subject to nonlinear boundary conditions, and in other areas. We refer the reader to [5] and [7].

There has been some recent work on this problem [1], [3], [4], [6] and the results are basically of two types. The first insists that $f(t)$ is smooth, i.e., $f \in W^{1,2}[0, T; H]$, while the second allows f to be an arbitrary element of $L^2[0, T; H]$, but requires that the nonlinear operator A have at most linear growth.

We will show that if A takes bounded subsets of H into bounded sets and f is in $L^\infty[0, T; H]$, then (1.1) has a unique local solution. By further restricting the kernel function $a(t)$ we are able to show the existence of global solutions. We are also able to show that by further restricting the growth of A, i.e., $|Ax|$ grows no faster than some polynomial in $|x|$, then, even for unbounded $f(t)$, (1.1) still has a unique solution. This result (Proposition 2.6) extends the results of [1], [4]. A result concerning the asymptotic behavior of the solution $u(t)$ is also included and an example shows this is best possible.

In §§II and III we state our results and give their proofs, respectively. In the last section we give several examples.

We will use the following notation throughout:

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$\langle \cdot, \cdot \rangle$, $|\cdot|$ will denote the inner product and norm of H, respectively,

$$L^p[0, T; H] = \left\{ f | f\colon [0, T] \to H, \int_0^T |f(s)|^p \, ds < \infty \right\}, \ 1 \leqslant p < \infty,$$

$$L^\infty[0, T; H] = \left\{ f | f\colon [0, T] \to H, \ \operatorname*{ess\,sup}_{0 < t < T} |f(t)| < \infty \right\},$$

(1.2) $$\langle u, v \rangle_{L^2[0,T;H]} = \int_0^T \langle u(t), v(t) \rangle \, dt,$$

$$\|u\|^2_{L^2[0,T;H]} = \langle u, u \rangle_{L^2[0,T;H]},$$

$$J_\lambda = (I + \lambda A)^{-1}, \qquad \lambda > 0,$$

$$A_\lambda = \lambda^{-1}[I - J_\lambda], \qquad \lambda > 0 \text{ (Yosida approximate of } A),$$

$A^0 x$ will denote the unique element in Ax of minimal norm.

II. Statement of results. We will assume that the kernel function $a(t)$ satisfies either of the following two conditions

(2.1)
(i) $a(0) > 0$.
(ii) $a(t)$ is absolutely continuous and a' is of bounded variation for $0 \leqslant t \leqslant T$.

(2.2) $a(t)$ is in $C[0, \infty; \mathbf{R}] \cap L^1_{\text{loc}}[0, \infty; \mathbf{R}]$ and is of positive type: this means that the map which sends $u(t) \to \int_0^t a(t - s)u(s)\, ds$ is nonnegative, i.e.,

(2.3) $$\int_0^T u(\tau) \int_0^\tau a(\tau - s) u(s) \, ds \, d\tau \geqslant 0, \quad \text{for } u \in L^2[0, T; \mathbf{R}].$$

A necessary and sufficient condition that $a(t)$ satisfy (2.3) is

(2.4) $$\liminf_{\sigma \to 0^+} \operatorname{Re} \hat{a}(\sigma + i\tau) \geqslant 0 \text{ for any real } \tau,$$

where

$$\hat{a}(s) = \int_0^\infty e^{-s\tau} a(\tau) \, d\tau \quad [\mathbf{8}].$$

Lemma 2.1 and Theorem 2.3 below summarize the tenchique developed by Londen [**6**].

Lemma 2.1 (Londen [**6**]). *Let $a(t)$ satisfy* (2.1). *Let u_n be a sequence in $L^2[0, T;\ H]$ such that $\|u_n\|_{L^2[0,T;H]}$ are uniformly bounded. Then either $\int_0^t a(t - s) u_n(s)\, ds$ converges uniformly on $[0, T]$ to zero as n tends to infinity, or there is a $\hat{t}$, $0 \leqslant \hat{t} \leqslant T$, such that*

(2.5) $$\limsup_{n \to \infty} \int_0^{\hat{t}} \left\langle \int_0^\tau a(\tau - s) u_n(s) \, ds, u_n(\tau) \right\rangle d\tau > 0.$$

Lemma 2.2 (Staffans [**9**]). *Let $a(t)$ satisfy* (2.2). *Then for all $u \in L^2[0, T;$ $H]$,*

$$(2.6)\qquad \left|\int_0^t a(t-s)u(s)\,ds\right|^2 \leqslant 2a(0)\int_0^t \left\langle \int_0^\tau a(\tau-s)u(s)\,ds, u(\tau)\right\rangle d\tau, \qquad 0 \leqslant t \leqslant T.$$

By u_λ we denote the unique $L^2[0, T; H]$ solution of

$$(2.7)\qquad u_\lambda(t) + \int_0^t a(t-s)A_\lambda u_\lambda(s)\,ds = f(t), \qquad \lambda > 0,$$

where $a(t)$ satisfies (2.1) or (2.2). Since the Yosida approximate A_λ of A is Lipschitz continuous it is clear that (2.7) has a unique solution for any f in $L^2[0, T; H]$.

THEOREM 2.3. *Let $a(t)$ satisfy* (2.1) *or* (2.2). *Suppose the u_λ and $A_\lambda u_\lambda$ defined by* (2.7) *are bounded in* $L^2[0, T; H]$ *independent of* λ. *Then* (1.1) *has a solution $u(t)$. That is, there is a pair of functions $u(t)$ and $w(t)$ both in* $L^2[0, T; H]$ *such that*

$$(2.8)\qquad u(t) + \int_0^t a(t-s)w(s)\,ds = f(t), \qquad w(t) \in Au(t) \text{ a.e.}$$

Moreover if $a(t)$ satisfies (2.1) *the pair u, w is unique or, if $a(t)$ satisfies* (2.2) *and A is strictly monotone then u is unique.*

PROPOSITION 2.4. *Let A be defined on all of H, and take bounded sets into bounded sets, i.e., $\bigcup_{|x|<r} Ax$ is bounded for any finite r. Let $a(t)$ satisfy* (2.1). *Then if $f \in L^\infty[0, T; H]$, there exists a $t_0 > 0$ such that u_λ and $A_\lambda u_\lambda$ defined by* (2.7) *are uniformly bounded in $L^\infty[0, t_0; H]$, and hence in $L^2(0, t_0; H)$.*

Theorem 2.3 and Proposition 2.4 imply the existence of a local solution to (1.1). Moreover this solution must also lie in $L^\infty[0, t_0; H]$, which implies that the solution to (1.1) may be continued as long as its supremum norm remains finite.

We remark that the requirement that A take bounded sets into bounded sets can be weakened to A being bounded in a neighborhood of $f(0)$, if $f(t)$ is continuous at $t = 0$. In this case our proof of Proposition 2.4 is still valid.

Our next results establish the existence of global solutions to (1.1).

PROPOSITION 2.5. *Let A be defined on all of H and take bounded sets into bounded sets. Let $a(t)$ satisfy* (2.2). *If $f \in L^\infty[0, T; H]$, then* (1.1) *has a unique solution on* $[0, T]$.

The above easily implies that if $f \in L^\infty_{\text{loc}}[0, \infty; H]$, then (1.1) has a unique solution in $L^\infty_{\text{loc}}[0, \infty; H]$.

PROPOSITION 2.6. *Let A have at most polynomial growth, that is, there is a constant c and a positive integer p such that*

$$(2.9)\qquad |v| \leqslant c\left(1 + |u|^p\right) \qquad \text{for every } (u,v) \in A.$$

Let $a(t)$ satisfy (2.2) *and assume $f \in L^{2p}[0, T; H]$. Then* (1.1) *has a unique solution on* $[0, T]$.

From the proof of Proposition 2.5 we derive the estimate

$$|u(t) - f(t)| \leqslant c\int_0^t |A^0 f(\tau)|\, d\tau. \tag{2.10}$$

Example 1 in §IV shows this is the best possible growth condition without further restrictions on (1.1). In particular f in $L^\infty[0, \infty; H]$ need not imply u in $L^\infty[0, \infty; H]$.

Arguing as in the proof of Proposition 2.5 the solutions of (1.1) can be shown to depend on f in a continuous manner. That is, the map which sends f into u is continuous from $L^\infty[0, T; H]$ onto $L^\infty[0, T; H]$.

III. Proofs.

THEOREM 2.3. Let $a(t)$ satisfy (2.1). Since the u_λ and $A_\lambda u_\lambda$ are bounded in $L^2[0, T; H]$ we may extract subsequences u_{λ_n} and $A_{\lambda_n} u_{\lambda_n}$ (which will be denoted by u_n and $A_n u_n$) which converge weakly to u and w respectively. It is easily seen that u and w satisfy (2.8). To show $w(t) \in Au(t)$ we first claim that

$$\int_0^t a(t - s) A_n u_n(s)\, ds \tag{3.1}$$

must converge uniformly on $[0, T]$. Suppose not, then there exist two subsequences $A_{n_k} u_{n_k}$ and $A_{m_k} u_{m_k}$ such that

$$\int_0^t a(t - s)\left[A_{n_k} u_{n_k}(s) - A_{m_k} u_{m_k}(s)\right] ds \tag{3.2}$$

does not converge uniformly to zero as k tends to infinity. By Lemma 2.1 there is a $\hat{t}$, $0 < \hat{t} < T$, such that

$$\begin{aligned}
0 &< \limsup_{k\to\infty} \int_0^{\hat{t}} \Big\langle A_{n_k} u_{n_k}(\tau) - A_{m_k} u_{m_k}(\tau), \\
&\qquad\qquad \int_0^\tau a(\tau - s)\left[A_{n_k} u_{n_k}(s) - A_{m_k} u_{m_k}(s)\right] ds \Big\rangle\, d\tau \\
&= \limsup_{k\to\infty} \int_0^{\hat{t}} \left\langle A_{n_k} u_{n_k}(\tau) - A_{m_k} u_{m_k}(\tau), u_{m_k}(\tau) - u_{n_k}(\tau)\right\rangle d\tau \qquad (3.3) \\
&\leqslant \limsup_{k\to\infty} \int_0^{\hat{t}} \big\langle A_{n_k} u_{n_k}(\tau) - A_{m_k} u_{m_k}(\tau), \\
&\qquad\qquad u_{m_k}(\tau) - J_{m_k} u_{m_k}(\tau) + J_{n_k} u_{n_k}(\tau) - u_{n_k}(\tau)\big\rangle\, d\tau
\end{aligned}$$

since $A_\lambda u \in AJ_\lambda u$ and A is monotone. This gives us

$$\begin{aligned}
0 &< \limsup_{k\to\infty} \int_0^{\hat{t}} \big\langle A_{n_k} u_{n_k}(\tau) - A_{m_k} u_{m_k}(\tau), \\
&\qquad\qquad \lambda_{m_k} A_{m_k} u_{m_k}(\tau) - \lambda_{n_k} A_{n_k} u_{n_k}(\tau)\big\rangle\, d\tau.
\end{aligned} \tag{3.4}$$

Since the $A_\lambda u_\lambda$'s are uniformly bounded in $L^2[0, T; H]$ and $\lambda_{n_k} \to 0$ as $k \to \infty$ we have that $0 < 0$. Thus (3.1) must converge uniformly to $\int_0^t a(t-s)w(s)\,ds$. From this and (2.7) we may conclude that u_n converges strongly to u. Now let $\mathcal{A}$ denote the usual extension of A to $L^2[0, T; H]$. We have that $u_n \in D(\mathcal{A})$ and $A_n u_n = \mathcal{A}_n u_n$. Let $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$. Then $\mathcal{A}_n x \to \mathcal{A}^0 x$, and

$$\begin{aligned} 0 &\leqslant \lim_{n\to\infty} \langle \mathcal{A}_n x - \mathcal{A}_n u_n, x - u_n \rangle_{L^2[0,T;H]} \\ &= \langle \mathcal{A}^0 x - w, x - u \rangle_{L^2[0,T;H]}. \end{aligned} \tag{3.5}$$

Since $\mathcal{A}$ is maximal monotone, (3.5) implies that

$$0 \leqslant \langle y - w, x - u \rangle_{L^2[0,T;H]} \tag{3.6}$$

for all $(x, y) \in \mathcal{A}$. Thus $(u, w) \in \mathcal{A}$, that is, $w(t) \in Au(t)$ a.e. cf. [**2**]. We have now shown that a solution to (1.1) exists. To see that u and w are unique, suppose that (x, y) is another solution pair. We then get

$$u(t) - x(t) + \int_0^t a(t-s)\left[w(s) - y(s)\right] ds = 0. \tag{3.7}$$

From Lemma 2.1 either the integral term is identically zero or as before we get from some $\hat{t}$ that

$$0 < \int_0^{\hat{t}} \langle w(\tau) - y(\tau), x(\tau) - u(\tau) \rangle \, d\tau, \tag{3.8}$$

which, from the monotonicity of A, is impossible. Thus $u(t) = x(t)$ and $w(t) = y(t)$ a.e.

To see that Theorem 2.3 is true when $a(t)$ satisfies (2.2) we refer the reader to Barbu [**1**, Theorem 1, p. 733].

PROPOSITION 2.4. Let $|a(t)| < M$, $0 \leqslant t \leqslant T$, $\|f\|_{L^\infty[0,T;H]} \leqslant M_1$, and M_2 be such that $|x| \leqslant 2M_1$ implies $|A^0 x| \leqslant M_2$. Then

$$\|u_\lambda\|_{L^\infty[0,t_0;H]} \leqslant 2M_1, \qquad t_0 = M_1/MM_2. \tag{3.9}$$

To prove (3.9) let t_1 be the largest t such that $\|u_\lambda\|_{L^\infty[0,t_1;H]} \leqslant 2M_1$. Suppose $t_1 < t_0$, then for $t \leqslant t_1$ we have

$$\begin{aligned} |u_\lambda(t)| &\leqslant |f(t)| + \int_0^t |a(t-s)|\,|A_\lambda u_\lambda(s)|\,ds \\ &\leqslant M_1 + M\int_0^t |A^0 u_\lambda(s)|\,ds < 2M_1. \end{aligned} \tag{3.10}$$

Since the integral term in (3.10) is a continuous function of t, we see that t_1 cannot be less than t_0. Thus on the interval $[0, t_0]$, the u_λ's are uniformly bounded and so are the $A_\lambda u_\lambda$'s since $|A_\lambda u_\lambda(t)| \leqslant |A^0 u_\lambda(t)|$, and A takes bounded sets into bounded sets.

PROPOSITION 2.5. Letting c denote a generic positive constant we have from Lemma 2.2 and (2.7) that

$$
\begin{aligned}
|u_\lambda(t) - f(t)|^2 &= \left|\int_0^t a(t-s)A_\lambda u_\lambda(s)\,ds\right|^2 \\
&\leqslant c\int_0^t \left\langle A_\lambda u_\lambda(\tau), \int_0^\tau a(\tau-s)A_\lambda u_\lambda(s)\,ds\right\rangle d\tau \\
&= c\int_0^t \langle A_\lambda u_\lambda(\tau), f(\tau) - u_\lambda(\tau)\rangle\,d\tau \qquad (3.11)\\
&\leqslant c\int_0^t \langle A_\lambda f(\tau), f(\tau) - u_\lambda(\tau)\rangle\,d\tau \\
&\leqslant c\int_0^t |A^0 f(\tau)|\,|f(\tau) - u_\lambda(\tau)|\,d\tau.
\end{aligned}
$$

Thus by an inequality similar to Gronwall's we have

$$|u_\lambda(t) - f(t)| \leqslant c\int_0^t |A^0 f(\tau)|\,d\tau. \tag{3.12}$$

Since $f \in L^\infty[0, T; H]$ and A takes bounded sets into bounded sets, the u_λ's are uniformly bounded in $L^\infty[0, T; H]$. Hence we have the u_λ's uniformly bounded in $L^2[0, T; H]$ and thus so are the $A_\lambda u_\lambda$'s. Proposition 2.5 now follows from Theorem 2.3.

PROPOSITION 2.6. As in Proposition 2.5 we have

$$\|u_\lambda - f\|_{L^2[0,T;H]} \leqslant cT\|A^0 f\|_{L^2[0,T;H]}. \tag{3.13}$$

As f is assumed to be in $L^{2P}[0, T; H]$ and A satisfies (2.9), the u_λ's are again uniformly bounded. Moreover from (2.7) we may show that the u_λ's are also uniformly bounded in $L^{2P}[0, T; H]$. Hence the $A_\lambda u_\lambda$'s are uniformly bounded in $L^2[0, T; H]$.

IV. Examples. Example 1 below shows that the growth estimate

$$|u(t) - f(t)| \leqslant c\int_0^t |A^0 f(\tau)|\,d\tau \tag{4.1}$$

obtained from (3.12) is the best possible.

EXAMPLE 1. Let $H = \mathbf{R}$, $Au = -1$, $a(t) = 1$, $f(t) = 0$. Equation (1.1) is now

$$u(t) + \int_0^t (-1)\,ds = 0. \tag{4.2}$$

The solution $u(t) = t$ shows that (4.1) is precise.

The next example demonstrates that if the forcing term is not in $L^\infty[0, T; H]$, then even for bounded nonlinear operators we will not get L^2 solutions. That is, (1.1) may have a solution but not both u and Au will be in $L^2[0, T; H]$.

EXAMPLE 2. Let $H = \mathbf{R}$, $Au = |u|u$, $a(t) = 1$, $f(t) = t^{-1/4}$. We then have from (1.1)

$$u(t) + \int_0^t |u(s)|u(s)\,ds = t^{-1/4}. \tag{4.3}$$

Since any solution of (4.3) cannot lie in $L^4[0, T; \mathbf{R}]$, we see that $Au = |u|u$ is not in $L^2[0, T; \mathbf{R}]$. Clearly this same type of argument holds if $Au = |u|^\alpha u$ for any positive α.

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