SYMMETRY FOR FINITE DIMENSIONAL HOPF ALGEBRAS

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Abstract. This note refines criteria given by R. G. Larson and M. E. Sweedler for a finite dimensional Hopf algebra to be a symmetric algebra, with applications to restricted universal enveloping algebras and to certain finite dimensional subalgebras of the hyperalgebra of a semisimple algebraic group in characteristic p.

Let $A$ be a finite dimensional associative algebra over a field $K$. Then $A$ is called Frobenius if there exists a nondegenerate bilinear form $f: A \times A \rightarrow K$ which is associative in the sense that $f(ab, c) = f(a, bc)$ for all $a, b, c \in A$ [3, Chapter IX]. $A$ is called symmetric if there exists a symmetric form of this type [3, §66]. For example, semisimple algebras and group algebras of finite groups are symmetric. We investigate here the extent to which finite dimensional Hopf algebras (with antipode) are symmetric; they are always Frobenius, thanks to the main theorem of [8].

1. Hopf algebras. In this section $H$ denotes a finite dimensional Hopf algebra over an arbitrary field $K$, with antipode $s$ and augmentation $\varepsilon: H \rightarrow K$. According to the main theorem of [8], existence of the antipode implies (and is implied by) the existence of a (nonsingular) left integral $\Lambda \in H$, which is unique up to scalar multiples. By definition, $\Lambda$ satisfies: $h\Lambda = \varepsilon(h)\Lambda$, for all $h \in H$. Equally well, $H$ has a right integral $\Lambda'$, unique up to scalar multiples. If $\Lambda'$ is proportional to $\Lambda$, $H$ is called unimodular.

With a left integral $\Lambda$ is associated a nondegenerate bilinear associative form $b$ on $H$ [8, §7]. As a result, $H$ is a Frobenius algebra. From the second corollary of Proposition 8 in [8], applied to the dual Hopf algebra (whose antipode has the same order as $s$), we obtain immediately:

Theorem 1. With notation as above, $b$ is symmetric if and only if $H$ is unimodular and $s^2 = 1$. In particular, if the latter conditions hold, then $H$ is symmetric.

We can apply this to the algebras $u_n$ ($n = 1, 2, \ldots$) defined in [6, Appendix U], [7]. These are finite dimensional Hopf subalgebras of the hyperalgebra $U_K$ of a simply connected, semisimple algebraic group $G$ over an algebraically
closed field $K$ of characteristic $p > 0$, with antipode of order 2.

**Corollary.** The algebras $u_n$ are symmetric.

**Proof.** $G$ acts naturally on $u_n$ as a group of Hopf algebra automorphisms (adjoint action) and therefore $g\Lambda$ is another left integral, if $g \in G$ and $\Lambda$ is a left integral in $u_n$; this must be of the form $\alpha(g)\Lambda$ for $\alpha(g) \in K^*$. Since $G = (G, G)$, it has no nontrivial homomorphisms into $K^*$, forcing $\alpha = 1$; therefore, $\Lambda$ is $G$-invariant. In turn, $\Lambda$ is invariant under the derived adjoint action of $u_n$, i.e., $\Lambda$ lies in the center of $u_n$ and is therefore also a right integral. Now $u_n$ is unimodular, with antipode of order 2, so the theorem applies.

Q.E.D.

**Remarks.** (1) W. J. Haboush [4] has recently been able to construct explicitly an integral for each $u_n$, and has used these to obtain a new family of “central differential operators” which are capable of separating the various Steinberg modules for $G$. His work inspired the present note, in particular the proof of the corollary.

(2) It follows from the corollary that the Cartan matrix $C$ of $u_n$, recording the composition factor multiplicities of principal indecomposable modules, is symmetric. Perhaps it is even of the form $C = D \cdot D$ for a suitable “decomposition matrix” $D$. This is the case when $n = 1$ [5], $u_1$ being the restricted universal enveloping algebra of the Lie algebra of $G$.

**Theorem 2.** If $H$ is symmetric, then $H$ is unimodular.

**Proof.** Let $f$ be a nondegenerate symmetric, associative bilinear form on $H$, and let $\Lambda$ (resp. $\Lambda'$) be a left (resp. right) integral. If $h \in \text{Ker} \, \varepsilon$, $f(h, \Lambda) = f(1, h\Lambda) = 0 = f(\Lambda'h, 1) = f(\Lambda', h)$. Since $f$ is symmetric, both $\Lambda$ and $\Lambda'$ lie in the orthogonal complement of $\text{Ker} \, \varepsilon$, which has dimension 1. So $\Lambda$ is proportional to $\Lambda'$. Q.E.D.

If $H$ is commutative or cocommutative, then automatically $s^2 = 1$ (cf. [8, p. 77]); in any event, $s$ has finite order [9]. But examples are known for which $s$ has higher (necessarily even) order. It is not clear whether such algebras can be unimodular without being symmetric.

### 2. Restricted enveloping algebras

In this section $K$ is a field of characteristic $p > 0$, $L$ a restricted Lie algebra (Lie $p$-algebra) over $K$, $u(L)$ its restricted universal enveloping algebra, which is a finite dimensional Hopf algebra with antipode of order 1 or 2. A. Berkson [1] showed directly that $u(L)$ is Frobenius by using a bilinear form $f$ defined as follows. Fix an ordered basis $x_1, \ldots, x_n$ of $L$, so the monomials $x_1^{i_1} \cdots x_n^{i_n}$ ($0 < i_j < p$) form a basis of $u(L)$. Let $\psi_0$ be the linear function taking value 1 at $u_0 = x_1^{p-1} \cdots x_n^{p-1}$ and value 0 at other monomials, and define $f(u, v) = \psi_0(uv)$. In turn, J. R. Schue proved:

**Theorem 3** [10]. The bilinear form $f$ is symmetric if and only if $\text{Tr}(\text{ad} \, x) = 0$
for all \( x \in L \). In particular, if the latter condition is satisfied, \( u(L) \) is symmetric.

Here \( \text{ad } x(y) = [x,y] \); the notation \( D_x \) is used in [10] and \( D(x) \) in [8]. This criterion for symmetry was applied in [5] to show that the algebra \( u_1 \) discussed above is symmetric. On the other hand, an example given in [8, p. 85] to show that \( u(L) \) need not be unimodular also fails (as it must, by Theorems 2 and 3) to meet Schue’s criterion.

The third corollary of Proposition 8 in [8] states that \( u(L) \) is unimodular if and only if \( \text{Tr}(\text{ad } x) = 0 \) for all \( x \in L \). We offer here a different proof, based on the following lemma.

**Lemma.** Let \( \Lambda \) be a left integral in \( u(L) \). If \( x_1, \ldots, x_n \) is any ordered basis of \( L \), and \( \Lambda \) is written as a linear combination of the monomials \( x_1^{i_1} \cdots x_n^{i_n} \), then \( u_0 = x_1^{i_n-1} \cdots x_1^{i_n-1} \) must occur with nonzero coefficient.

**Proof.** Suppose the contrary. Define the degree of \( x_1^{i_1} \cdots x_n^{i_n} \) to be \( \Sigma i_j \), and write \( \Lambda = \Lambda_1 + \Lambda_2 \), where \( \Lambda_1 \) contains all monomials of the highest degree \( d \) occurring (so \( d < n(p-1) \)). If we change basis by permuting the \( x_j \), the new expression for \( \Lambda \) will be of the form \( \Lambda_1' + \Lambda_3 \), where \( \Lambda_1' \) is obtained from \( \Lambda_1 \) by permuting the \( x_j \) and the monomials in \( \Lambda_3 \) are again of degree \( < d \) (cf. the proof of Lemma 1 in [10]). We may therefore assume that not all monomials occurring in \( \Lambda_1 \) involve the factor \( x_1^{i_n-1} \).

The augmentation for \( u(L) \) maps all \( x_j \) to 0, so by definition of left integral, \( 0 = x_1 \Lambda = x_1 \Lambda_1 + x_1 \Lambda_2 \). Because of the \( p \)-structure, \( x_1^{p} x_1^{j_2} \cdots x_1^{j_n} \) can be rewritten in \( u(L) \) as a linear combination of monomials having degrees \( < p + \Sigma j_i \) (sum over \( j \neq 1 \)). In particular, \( x_1 \Lambda_2 \) involves only monomials of degree \( < d \), while \( x_1 \Lambda_1 \) involves such monomials along with one or more linearly independent monomials of degree \( d + 1 \) (corresponding to monomials in \( \Lambda_3 \) not involving \( x_1^{i_n-1} \)). This is clearly impossible. Q.E.D.

We remark that the integral constructed by Haboush [4] for the algebra \( u_1 \) illustrates this lemma very nicely.

**Theorem 4.** If \( u(L) \) is a symmetric algebra, then \( \text{Tr}(\text{ad } x) = 0 \) for all \( x \in L \).

**Proof.** Since \( u(L) \) is symmetric, it is unimodular (Theorem 2), so any integral \( \Lambda \) lies in the center of \( u(L) \). Choose an ordered basis \( x_1, \ldots, x_n \) of \( L \), and define the form \( f \) as above, relative to the basis of \( u(L) \) consisting of monomials. By the lemma, we may (after multiplying \( \Lambda \) by a nonzero scalar) write \( \Lambda = u_0 + u_1 \), where \( u_1 \) is a linear combination of monomials of degrees \( < n(p-1) \). With \( \varphi_0 \) as above, it follows from Lemma 1 of [10] (cf. proof of theorem) that \( \varphi_0(u_1 x) = \varphi_0(x u_1) \) for all \( x \in L \). On the other hand, Lemma 3 of [10] says that \( \varphi_0(u_0 x) = \varphi_0(x u_0) + \text{Tr}(\text{ad } x) \) for \( x \in L \). Since \( \Lambda x = x \Lambda \), it follows at once that \( \text{Tr}(\text{ad } x) = 0 \). Q.E.D.

Theorems 3 and 4, combined with Theorems 1 and 2, show that \( u(L) \) is unimodular if and only if \( \text{Tr}(\text{ad } x) = 0 \) for all \( x \in L \), as stated in [8]. This is analogous to the classical criterion for a Lie group \( G \) to be unimodular:
\[ \det(\text{Ad } g) = 1 \quad \text{for all } g \in G \quad [2, \text{Chapter III, 3, no. 16, Corollary to Proposition 55}]. \]

**References**

4. W. J. Haboush, (manuscript).

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