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THE KNAPP-STEIN DIMENSION THEOREM FOR p-ADIC GROUPS[1]

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ABSTRACT. Knapp and Stein have proved for semisimple Lie groups that the dimension of the commuting algebra of an induced tempered representation equals the index of a certain reflection group in a larger group. A precise analogue of their result is stated and proved in this paper for p-adic groups.

The purpose of this paper is to prove for p-adic groups the analogue of a theorem due to Knapp and Stein [3] in the case of real semisimple Lie groups. The Knapp-Stein theorem has precisely–mutatis mutandis–the same statement as we give below. Our proof, which depends upon the Harish-Chandra commuting algebra theorem [**4**, Theorem 5.5.3.2], carries over, with only slight changes, to the real case too.

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1. Some terminology. Let Ω be a nonarchimedean local field and $\mathbf{G}$ a connected reductive Ω-group. Let G denote the group of Ω-points of $\mathbf{G}$. In this paper we employ the terminology and notations of [**2**] and [**4**].

Fix a minimal p-pair (P_0, A_0) $(P_0 = M_0N_0)$ of G and an A_0-good maximal compact subgroup K of G. Let (P, A) $(P = MN)$ be a semistandard p-pair of G. Let $\mathfrak{a}^*$ denote the real Lie algebra of A. Let W denote the factor group $N_G(A)/M$. Assume that $\mathfrak{a}^*$ has a W-invariant scalar product defined on it. Let $\Sigma_r = \Sigma_r(P, A)$ denote the set of positive reduced A-roots, $\Sigma^0(P, A)$ the subset consisting of the simple A-roots.

Let $\sigma \in \omega \in \mathcal{E}_2(M)$. Let $W(\omega) = \{s \in W | \omega^s = \omega\}$. Let $\mu(\omega\colon \nu)$ $(\nu \in \mathfrak{a}^*)$ denote the Harish-Chandra function associated to ω and G [**2**, Theorem 20], [**4**, §5.4.3]. It is proved in [**4**, Corollary 5.4.3.3] (cf. [**2**, Theorem 24]) that, with $c > 0$,

$$c\mu(\omega\colon \nu) = \prod_{\alpha \in \Sigma_r} \mu_\alpha(\omega\colon \nu),$$

where $\mu_\alpha(\omega\colon \nu)$ is the Harish-Chandra function associated to ω and $M_\alpha = Z_G(A_\alpha)$ (A_α is the maximal subtorus of A in the kernel of the root character ξ_α). A root $\alpha \in \Sigma_r$ is called *ω-special* if $\mu_\alpha(\omega\colon 0) = 0$. If α is ω-special, then

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there is a reflection $s_\alpha \in W(\omega)$. Let $\Sigma'' = \pm\{\alpha \in \Sigma_r\colon \alpha\ \omega\text{-special}\}$. Then Σ'' is a root system in a subspace of $\mathfrak{a}^*$ [**1**, VI, §2, Proposition 9]. We write $W''(\omega)$ for the Weyl group of this root system; $W''(\omega)$ is the subgroup of $W(\omega)$ generated by the set $\{s_\alpha\colon \alpha\ \omega\text{-special}\}$.

Let $C_M^G(\omega)$ denote the class of the induced representation $\pi_{P,\omega} = \operatorname{Ind}_P^G(\delta_P^{1/2}\sigma)$. Then $C_M^G(\omega)$ is unitary and independent of the choice of $P \in \mathcal{P}(A)$ or ω in a W-orbit.

2. The theorem. In the following we assume that the c-functions and ${}^0c$-functions, as well as the space $L(\omega, P)$, are associated to a fixed smooth unitary double representation of K which satisfies associativity conditions.

THEOREM. *The commuting algebra of the class $C_M^G(\omega)$ has dimension* $[W(\omega)\colon W''(\omega)]$.

PROOF. Harish-Chandra's commuting algebra theorem implies that, for any $P \in \mathcal{P}(A)$, the mapping $s \mapsto {}^0c_{P|P}(s\colon \omega)$, a homomorphism from $W(\omega)$ to the group of unitary automorphisms of the algebra $L(\omega, P)$, may be regarded as a mapping onto a set of generators for the commuting algebra of $\operatorname{Ind}_P^G(\delta_P^{1/2}\sigma)$ $(\sigma \in \omega)$. We prove the theorem in two steps: (1) ${}^0c_{P|P}(s\colon \omega)$ is the identity on $L(\omega, P)$ when $s \in W''(\omega)$; (2) the dimension of the commuting algebra is at least $[W(\omega)\colon W''(\omega)]$.

For (1) it is enough to show that ${}^0c_{P|P}(s\colon \omega) = I$ whenever s is the reflection s_α associated to an ω-special root α. Given any such α, we may choose $P_1 \in \mathcal{P}(A)$ such that $\alpha \in \Sigma^0(P_1, A)$. It is enough to show that

$$ {}^0c_{P_1|P_1}(s\colon \omega) = I, $$

since

$$ {}^0c_{P|P}(s\colon \omega) = {}^0c_{P|P_1}(1\colon \omega)\, {}^0c_{P_1|P_1}(s\colon \omega)\, {}^0c_{P_1|P}(1\colon \omega) \quad \text{and} $$
$$ I = {}^0c_{P|P}(1\colon \omega) = {}^0c_{P|P_1}(1\colon \omega)\, {}^0c_{P_1|P}(1\colon \omega); $$

both relations follow from the general transformation formulas for the ${}^0c$-functions [**2**, §§11–12], [**4**, §5.2.4].

Thus, without loss of generality, assume that $\alpha \in \Sigma^0(P, A)$. Let A_α and M_α be as before. Then $P \cap M_\alpha = {}^*P_\alpha$ is a maximal parabolic subgroup of M_α. Since $\mu_\alpha(\omega\colon 0) = 0$, the representation $\operatorname{Ind}_{{}^*P_\alpha}^{M_\alpha}(\delta_{{}^*P_\alpha}^{1/2}\sigma)$ is irreducible, so ${}^0c_{{}^*P_\alpha|{}^*P_\alpha}(s_\alpha\colon \omega) = I_{L(\omega, {}^*P_\alpha)}$. On the other hand, by [**4**, Theorem 5.3.5.3(4)],

$$ {}^0c_{P|P}(s_\alpha\colon \omega) = {}^0c_{{}^*P_\alpha|{}^*P_\alpha}(s_\alpha\colon \omega)\big|_{L(\omega, P)}, $$

so ${}^0c_{P|P}(s\colon \omega) = I_{L(\omega, P)}$ for all $s \in W''(\omega)$, as required.

To prove (2) we shall argue as follows. Let $\pi_{P,\omega} = \operatorname{Ind}_P^G(\delta_P^{1/2}\sigma)$ act in a vector space $\mathcal{H}$. Consider the tempered Jacquet module $\overline{\mathcal{H}} = {}_w(\mathcal{H}/\mathcal{H}(\overline{P}))$ associated to $\pi_{P,\omega}$, with $\overline{\pi}_{P,\omega}$ the representation of M on $\overline{\mathcal{H}}$. It is known [**4**, Theorem 5.4.1.1] that $\overline{\mathcal{H}}$ has a composition series of length $[W(G/A)]$, whose composition factors, counted with multiplicities, are $\{\delta_P^{1/2}\omega^s\}_{s \in W(G/A)}$. Furthermore, it follows from the fact that discrete series are projectives in the category of tempered modules (with a fixed central exponent) that $\overline{\mathcal{H}}$ is a direct sum of isotypic submodules. Let $\overline{\mathcal{H}}(\omega)$ be the submodule all of whose components are of class $\delta_P^{1/2}\omega$. The composition series for $\overline{\mathcal{H}}(\omega)$ has length $[W(\omega)]$. The Frobenius reciprocity theorem [**4**, Theorem 1.7.10] implies that $\delta_P^{1/2}\omega$ occurs as a quotient in $\overline{\mathcal{H}}(\omega)$ a number of times equal to the dimension of the commuting algebra of $C_M^G(\omega)$. Thus, to prove (2), it is sufficient to show that $\overline{\mathcal{H}}(\omega)$ contains $\delta_P^{1/2}\omega$ as a quotient at least $[W(\omega)\colon W''(\omega)]$ times. For this, it is obviously sufficient to show that the multiplicity of the central character $\delta_P^{1/2}\chi_\omega$ in $\overline{\mathcal{H}}(\omega)$ is no greater than $[W''(\omega)]$.

We shall prove, instead, an equivalent fact involving the Eisenstein integral and the weak constant term. Let $\psi \in L(\omega, P)$ and consider the Eisenstein integral $E(P : \psi : \nu)$. The weak constant term ${}_wE_P(P : \psi : \nu)$ is holomorphic in a neighborhood U of $\mathfrak{a}^*$ [**4**, Corollary 5.3.3.5]. For $\nu \in U$ in general position we may write

$$ {}_wE_P(P : \psi : \nu) = \sum_{s \in W(G/A)} c_{P|P}(s : \omega : \nu)\psi\chi_{s\nu}. $$

For any $s \in W(G/A)$, the function

$$ c_{P|P}(s : \omega : \nu) = s c_{P^{s^{-1}}|P}(1 : \omega : \nu) = s \prod_{\alpha \in \Sigma_r(P,A)} c_\alpha^{\pm}(1 : \omega : \nu), $$

where each function $c_\alpha^+(1 : \omega : \nu)$ or $c_\alpha^-(1 : \omega : \nu)$ is a c-function associated to a pair (M_α, M) in which M_α is a reductive subgroup of G containing $(P \cap M_\alpha, A)$ as a maximal p-pair [**4**, §5.4.3]. Each function $c_\alpha^{\pm}$ is essentially a meromorphic function of a single complex variable, holomorphic for all $\nu \in U$, unless α is an ω-special root; if α is an ω-special root, then the hyperplane H_α passing through $\nu = 0$ and orthogonal to α is singular for $c_\alpha^{\pm}$. This implies that the function $c_{P|P}(s : \omega : \nu)$ is holomorphic on $U - \bigcup_{\alpha \in \Sigma''} H_\alpha$.

We claim that, to prove (2), it is sufficient to show that the function

$$ \Phi(s_0, \nu) = \sum_{s \in W''(\omega)} c_{P|P}(ss_0 : \omega : \nu)\psi\chi_{ss_0\nu} $$

is holomorphic at $\nu = 0$ for any $s_0 \in W(\omega)$. If this is so, then one can show exactly as in [**4**, §§5.3.2–3] (and we shall *not* give the details here) that $\prod_{t \in W''(\omega)} (\chi_{ts_0\nu}(a) - \rho(a))\Phi(s_0, \nu)$ is identically zero near $\nu = 0$ and, as a consequence, that the multiplicity of the exponent χ_ω is no greater than $[W''(\omega)]$. However, by [**4**, Corollary 3.2.5(3)], the multiplicity of the exponent χ_ω related to the constant term is the same as the multiplicity of $\delta_P^{1/2}\chi_\omega$ in the Jacquet space. Thus, it follows easily that, since $\delta_P^{1/2}\omega$ occurs $[W(\omega)]$ times in the composition series of $\overline{\mathcal{H}}(\omega)$, $\delta_P^{1/2}\omega$ actually occurs as a quotient at least $[W(\omega)\colon W''(\omega)]$ times, as required.

Let us show that $\Phi(s_0, \nu)$ is holomorphic at $\nu = 0$. It is enough to check this for any $\psi \in L(\omega, P)$. As is well known, we may (and do) choose ψ such that $E(P\colon \psi\colon \nu) = E(P\colon \psi\colon s\nu)$ for all $s \in W(\omega)$ and $\nu \in \mathfrak{a}^*$. Observe that, in this case, $c_{P|P}(s:\omega:tv)\psi\chi_{stv} = c_{P|P}(st:\omega:\nu)\psi\chi_{st\nu}$ for all $s, t \in W(\omega)$ and $\nu \in \mathfrak{a}^*$, so $\Phi(1, s_0\nu) = \Phi(s_0, \nu)$. Thus, it is sufficient to check that $\Phi(1, \nu)$ is holomorphic at $\nu = 0$.

We shall need the fact that the weak constant term takes its image in the direct sum $\bigoplus_{s\in W/W(\omega_\nu)} \mathcal{Q}(M, \tau_M)_{\omega_{s\nu}^\cdot}$. This is proved in the supercuspidal case in [4, Corollary 5.4.4.6]; the proof in the present case is exactly the same and depends upon the fact, used above, that discrete series are projectives in the category of tempered admissible modules. As a consequence, any term

$$E_{P,\omega_{\nu_0}}(P:\psi:\nu) = \sum_{s\in W(\omega_{\nu_0})} E_{P,s}(P:\psi:\nu)$$

is holomorphic in a neighborhood of $\nu = \nu_0$.

We have already observed that the singularities of $\Phi(1, \nu)$, if there are any, lie in $\bigcup H_\alpha$ $(\alpha \in \Sigma'')$. It follows easily from the Weierstrass Preparation Theorem that a nonempty zero set of a holomorphic function defined in an open set U of a complex space is a union of hypersurfaces in U. Therefore, it is sufficient, in order to show that $\Phi(1, \nu)$ is holomorphic at $\nu = 0$, to show that the singularities lie in a subset of codimension at least two.

Let $\alpha \in \Sigma''$ and $\nu_0 \in H_\alpha - \bigcup_{\alpha'\neq\alpha} H_{\alpha'}$. We shall show that $\Phi(1, \nu)$ is holomorphic at $\nu = \nu_0$. To see this, note first that $W(\omega_{\nu_0}) \cap W''(\omega_0) = \{1, s_\alpha\}$, which follows from well-known properties of Weyl groups. We may choose representatives $s_1, \ldots, s_r \in W''(\omega) \setminus W(\omega)$ such that s_i and $s_\alpha s_i$ fix H_α for all $i = 1, \ldots, r$. There is a neighborhood V of ν_0 on which

$$E_{P,\omega_{\nu_0}}(P:\psi:\nu) = \sum_{i=1}^{r} \left(c_{P|P}(s_i:\omega:\nu)\psi\chi_{s_i\nu} + c_{P|P}(s_\alpha s_i:\omega:\nu)\psi\chi_{s_\alpha s_i\nu}\right)$$

is holomorphic. For all $\nu \in V \cap H_\alpha$ and $i = 1, \ldots, r$.

$$\begin{aligned} c_{P|P}(s_i:\omega:\nu)\psi\chi_{s_i\nu} &+ c_{P|P}(s_\alpha s_i:\omega:\nu)\psi\chi_{s_\alpha s_i\nu} \\ &= c_{P|P}(1:\omega:\nu)\psi\chi_\nu + c_{P|P}(s_\alpha:\omega:\nu)\psi\chi_{s_\alpha\nu}, \end{aligned}$$

from which it follows that $c_{P|P}(1:\omega:\nu)\psi\chi_\nu + c_{P|P}(s_\alpha:\omega:\nu)\psi\chi_{s_\alpha\nu}$ and, hence, $\Phi(1, \nu)$ is holomorphic near $\nu = \nu_0$. We conclude that $\Phi(1, \nu)$ is, in fact, holomorphic at $\nu = 0$. This proves the theorem.

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