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ON DENSITY OF ALGEBRAS WITH MINIMAL INVARIANT OPERATOR RANGES

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ABSTRACT. Let $\mathfrak{A}$ be an arbitrary subalgebra of $\mathfrak{B}(\mathfrak{H})$ and let $\mathfrak{M}$ be a dense operator range invariant under $\mathfrak{A}$ such that every nonzero operator range invariant under $\mathfrak{A}$ contains $\mathfrak{M}$. Then the closure of $\mathfrak{A}$ in the strong operator topology is $\mathfrak{B}(\mathfrak{H})$.

Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}(\mathfrak{H})$, the algebra of all (bounded, linear) operators on the complex Hilbert space $\mathfrak{H}$. No topological closure assumptions are made on $\mathfrak{A}$. The transitive algebra problem (cf. [7, p. 138]) can be stated as follows: if the only closed subspaces of $\mathfrak{H}$ invariant under (all members of) $\mathfrak{A}$ are $\{0\}$ and $\mathfrak{H}$, is $\mathfrak{A}$ strongly dense in $\mathfrak{B}(\mathfrak{H})$? Foiaş [5] showed that the answer is affirmative if the hypothesis is strengthened by substituting "operator ranges" for "closed subspaces." By an operator range is meant the range of an operator from $\mathfrak{H}$ into $\mathfrak{H}$; Foiaş sometimes uses the nomenclature "para-closed subspaces" for operator ranges.

Operator ranges have proven to be useful tools in dealing with questions related to reductivity and reflexivity of operator algebras. The results of Azoff [1] and Douglas and Foiaş [3] are examples. For an extension of the above-quoted result of Foiaş see [6].

In this note we prove strong density for those transitive algebras whose lattice of invariant operator ranges has a simple property; the nonzero elements of the lattice have a nonzero lower bound. Examples of such algebras are abundant; for a simple example let K be an injective operator $\mathfrak{H}$ with dense range (but $K\mathfrak{H} \neq \mathfrak{H}$ to make the example nontrivial). Let $\mathfrak{A}$ be the right ideal $K\mathfrak{B}(\mathfrak{H})$ of $\mathfrak{B}(\mathfrak{H})$. Then $K\mathfrak{H}$ is certainly invariant under $\mathfrak{A}$ and every nonzero operator range, in fact every nonzero linear manifold, invariant under $\mathfrak{A}$ must include $K\mathfrak{H}$.

We shall need the following lemmas.

LEMMA 1. *Let K be an injective operator whose range is a minimal, dense, invariant operator range for the algebra $\mathfrak{A}$. Then there is an algebra isomorphism ϕ of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ such that*

(i) $AK = K\phi(A)$ *for all* $A \in \mathfrak{A}$,

(ii) $\phi(\mathfrak{A})$ *is strongly dense in* $\mathfrak{B}(\mathfrak{H})$, *and*

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(iii) *if* $K\mathcal{H}$ *is contained in all nonzero invariant operator ranges of* $\mathfrak{A}$, *then the only operators L satisfying the equation* $AL = L\phi(A)$ *for all A in* $\mathfrak{A}$ *are scalar multiples of K.*

PROOF. (i) For each $A \in \mathfrak{A}$, $AK\mathcal{H} \subseteq K\mathcal{H}$ and thus there is an operator B with $AK = KB$; (see, e.g., [2]). The fact that B depends uniquely and isomorphically on A is easy to verify.

(ii) To verify strong density of $\phi(\mathfrak{A})$ it suffices to show that $\phi(\mathfrak{A})$ has no invariant operator ranges other than $\{0\}$ and $\mathcal{H}$ (by Foiaş [5]). Let M be a nonzero operator whose range is invariant under $\phi(\mathfrak{A})$. Assume, with no loss of generality, that M is injective. (If S is an isometry of $\mathcal{H}$ onto $\mathcal{H}/M^{-1}\{0\}$, then MV is injective and has the same range as M.) Thus $\phi(A)M = M\psi(A)$ for all $A \in \mathfrak{A}$, where ψ is an algebra homomorphism. Then $AKM = K\phi(A)M = KM\psi(A)$, which implies that $KM\mathcal{H}$ is invariant under $\mathfrak{A}$. By the minimality of $K\mathcal{H}$, $K\mathcal{H} = KM\mathcal{H}$; since K is injective, we obtain $M\mathcal{H} = \mathcal{H}$.

(iii) First note that $AL = L\phi(A)$ implies that either L is zero or it is injective, because if $Lx = 0$ with $x \neq 0$, then $L\phi(\mathfrak{A})x = \mathfrak{A}Lx = 0$. Since $\phi(\mathfrak{A})$ is strongly dense in $\mathcal{B}(\mathcal{H})$, this implies $L\mathcal{H} = 0$.

Now the equation $AL = L\phi(A)$ also implies that $L\mathcal{H}$ is invariant under $\mathfrak{A}$ and thus there is an operator M with $LM = K$. Then

$$LM\phi(A) = ALM = L\phi(A)M$$

or, assuming $L \neq 0$, $M\phi(A) = \phi(A)M$. Since $\phi(\mathfrak{A})$ is dense in $\mathcal{B}(\mathcal{H})$, M commutes with $\mathcal{B}(\mathcal{H})$ and must be a scalar operator.

Before stating the next lemma we fix some notation and terminology. For $A \in \mathcal{B}(\mathcal{H})$ we denote by $A^{(n)}$ the direct sum of n copies of A acting on $\mathcal{H}^{(n)}$, the direct sum of n copies of $\mathcal{H}$; for an algebra $\mathfrak{A}$ of operators, $\mathfrak{A}^{(n)}$ will stand for $\{A^{(n)}: A \in \mathfrak{A}\}$. A *graph subspace* for $\mathfrak{A}^{(n)}$ is a closed subspace of $\mathcal{H}^{(n)}$, invariant under $\mathfrak{A}^{(n)}$, which has the form

$$\{x \oplus T_1x \oplus \cdots \oplus T_{n-1}x: x \in \mathcal{D}\},$$

where $\mathcal{D}$ is a dense linear manifold in $\mathcal{H}$ and, for each i, T_i is a linear transformation from $\mathcal{D}$ into $\mathcal{H}$. Each T_i is called a *graph transformation* for $\mathfrak{A}$. This definition implies, in particular, that $\mathcal{D}$ and the $T_i\mathcal{D}$ are invariant under $\mathfrak{A}$.

The following lemma due to Arveson, will be used; for a proof see [7, p. 143].

LEMMA 2. *Let* $\mathfrak{A}$ *be an algebra with the property that every graph transformation for* $\mathfrak{A}$ *is a scalar multiple of the identity on* $\mathcal{H}$. *Then* $\mathfrak{A}$ *is strongly dense in* $\mathcal{B}(\mathcal{H})$.

(Note that the hypothesis on $\mathfrak{A}$ implies that $\mathfrak{A}$ has no closed invariant subspaces except $\{0\}$ and $\mathcal{H}$.)

THEOREM. *Let* $\mathfrak{A}$ *be a subalgebra of* $\mathcal{B}(\mathcal{H})$. *Assume that* $\mathfrak{A}$ *has a dense,* *invariant operator range contained in every nonzero invariant operator range. Then $\mathfrak{A}$ is strongly dense in $\mathfrak{B}(\mathfrak{H})$.*

PROOF. Let $K \in \mathfrak{B}(\mathfrak{H})$ such that $K\mathfrak{H}$ is the lower bound of the nonzero invariant operator ranges as hypothesized. The operator K can also be assumed to be injective. Now $AK = K\phi(A)$ for all $A \in \mathfrak{A}$, where ϕ is the isomorphism given by Lemma 1.

Let n be any positive integer and let $\mathfrak{M}$ be a graph subspace for $\mathfrak{A}^{(n)}$. Then $\mathfrak{M}$ can be considered as the graph of a closed linear transformation T from a dense domain $\mathfrak{D}$ in $\mathfrak{H}$ into $\mathfrak{H}^{(n-1)}$ defined by $Tx = T_1x \oplus \cdots \oplus T_{n-1}x$, $x \in \mathfrak{D}$. Now $\mathfrak{D} = C_1\mathfrak{H}$, where C_1 is a bounded, injective operator on $\mathfrak{H}$ (see, e.g., [4]). Thus

$$\mathfrak{M} = \{C_1y \oplus TC_1y : y \in \mathfrak{H}\};$$

since TC_1 is closed, it follows from the closed-graph theorem that it is bounded. Hence T_iC is bounded, for each i, because $T_iC = P_iTC$, where P_i is a projection. We can now represent $\mathfrak{M}$ as

$$\{C_1x \oplus C_2x \oplus \cdots \oplus C_nx : x \in \mathfrak{H}\},$$

where each C_i is a bounded operator and where C_1 is, furthermore, injective.

The inclusion $A^{(n)}\mathfrak{M} \subseteq \mathfrak{M}$ implies that for each x and each $A \in \mathfrak{A}$ there exists a y with $AC_ix = C_iy$ for $i = 1, \ldots, n$. The injectivity of C_1 implies that y is unique, and it follows that there is an algebra isomorphism ψ of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ such that $AC_i = C_i\psi(A)$ for all A and all i. In particular, $C_1\mathfrak{H}$ is invariant under $\mathfrak{A}$ and thus $C_1B = K$ for some operator B. Thus, by the equation $AK = K\phi(A)$,

$$C_1B\phi(A) = AC_1B = C_1\psi(A)B$$

and, therefore, $B\phi(A) = \psi(A)B$. Now, for each i,

$$AC_iB = C_i\psi(A)B = C_iB\phi(A),$$

and Lemma 1 implies that $C_iB = \alpha_iK = \alpha_iC_1B$ for some scalar α_i. Thus

$$y \oplus \alpha_2y \oplus \cdots \oplus \alpha_ny \in \mathfrak{M}$$

for every $y \in K\mathfrak{H}$. Since $K\mathfrak{H}$ is dense in $\mathfrak{H}$,

$$\mathfrak{M} = \{y \oplus \alpha_2y \oplus \cdots \oplus \alpha_ny : y \in \mathfrak{H}\}.$$

We have shown that every T_i is a scalar multiple of the identity on $\mathfrak{H}$. The strong density of $\mathfrak{A}$ in $\mathfrak{B}(\mathfrak{H})$ now follows from Lemma 2.

COROLLARY 1. *If in the lattice of invariant operator ranges for $\mathfrak{A}$ the nonzero elements have a nonzero lower bound, then either $\mathfrak{A}$ is strongly dense in $\mathfrak{B}(\mathfrak{H})$ or it has a nontrivial closed, invariant subspace.*

PROOF. If this lower bound represents a dense range, we apply the above theorem; otherwise, the closure of this range is a nontrivial invariant subspace.

COROLLARY 2. *If $\mathfrak{A}$ is as in Corollary* 1 *and if $\mathfrak{M}$ is the lower bound for* *nonzero, invariant operator ranges, then the restriction* $\mathfrak{A}|\overline{\mathfrak{M}}$ *of* $\mathfrak{A}$ *of the closure of* $\mathfrak{M}$ *is dense in* $\mathfrak{B}(\overline{\mathfrak{M}})$.

PROOF. It is easily verified that $\mathfrak{M}$ is the lower bound of the nonzero operator ranges invariant under $\mathfrak{A}|\overline{\mathfrak{M}}$; thus the theorem is applicable.

QUESTIONS. One natural question poses itself: Can the lattice-theoretic hypothesis of the above results be weakened to require the mere existence of a minimal nonzero element in this lattice? What about the more special case where every nonzero member of the lattice is assumed to contain a minimal nonzero element?

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