ON DENSITY OF ALGEBRAS  
WITH MINIMAL INVARIANT OPERATOR RANGES  

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ABSTRACT. Let $\mathfrak{A}$ be an arbitrary subalgebra of $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{M}$ be a dense operator range invariant under $\mathfrak{A}$ such that every nonzero operator range invariant under $\mathfrak{A}$ contains $\mathfrak{M}$. Then the closure of $\mathfrak{A}$ in the strong operator topology is $\mathcal{B}(\mathcal{H})$.

Let $\mathfrak{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra of all (bounded, linear) operators on the complex Hilbert space $\mathcal{H}$. No topological closure assumptions are made on $\mathfrak{A}$. The transitive algebra problem (cf. [7, p. 138]) can be stated as follows: if the only closed subspaces of $\mathcal{H}$ invariant under (all members of) $\mathfrak{A}$ are $\{0\}$ and $\mathcal{H}$, is $\mathfrak{A}$ strongly dense in $\mathcal{B}(\mathcal{H})$? Foiaş [5] showed that the answer is affirmative if the hypothesis is strengthened by substituting “operator ranges” for “closed subspaces.” By an operator range is meant the range of an operator from $\mathcal{H}$ into $\mathcal{H}$; Foiaş sometimes uses the nomenclature “para-closed subspaces” for operator ranges.

Operator ranges have proven to be useful tools in dealing with questions related to reductivity and reflexivity of operator algebras. The results of Azoff [1] and Douglas and Foiaş [3] are examples. For an extension of the above-quoted result of Foiaş see [6].

In this note we prove strong density for those transitive algebras whose lattice of invariant operator ranges has a simple property; the nonzero elements of the lattice have a nonzero lower bound. Examples of such algebras are abundant; for a simple example let $K$ be an injective operator $\mathcal{H}$ with dense range (but $K\mathcal{H} \neq \mathcal{H}$ to make the example nontrivial). Let $\mathfrak{A}$ be the right ideal $K\mathcal{B}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$. Then $K\mathcal{H}$ is certainly invariant under $\mathfrak{A}$ and every nonzero operator range, in fact every nonzero linear manifold, invariant under $\mathfrak{A}$ must include $K\mathcal{H}$.

We shall need the following lemmas.

**Lemma 1.** Let $K$ be an injective operator whose range is a minimal, dense, invariant operator range for the algebra $\mathfrak{A}$. Then there is an algebra isomorphism $\phi$ of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$ such that

(i) $AK = K\phi(A)$ for all $A \in \mathfrak{A},$

(ii) $\phi(\mathfrak{A})$ is strongly dense in $\mathcal{B}(\mathcal{H})$, and
(iii) If \( K \mathcal{H} \) is contained in all nonzero invariant operator ranges of \( \mathfrak{A} \), then the only operators \( L \) satisfying the equation \( AL = L \phi(A) \) for all \( A \) in \( \mathfrak{A} \) are scalar multiples of \( K \).

**Proof.** (i) For each \( A \in \mathfrak{A} \), \( AK \mathcal{H} \subseteq K \mathcal{H} \) and thus there is an operator \( B \) with \( AK = KB \); (see, e.g., [2]). The fact that \( B \) depends uniquely and isomorphically on \( A \) is easy to verify.

(ii) To verify strong density of \( \phi(\mathfrak{A}) \) it suffices to show that \( \phi(\mathfrak{A}) \) has no invariant operator ranges other than \( \{0\} \) and \( \mathcal{H} \) (by Foiaş [5]). Let \( M \) be a nonzero operator whose range is invariant under \( \phi(\mathfrak{A}) \). Assume, with no loss of generality, that \( M \) is injective. (If \( S \) is an isometry of \( \mathcal{H} \) onto \( \mathcal{H} / M^{-1}(0) \), then \( MV \) is injective and has the same range as \( M \).) Thus \( \phi(A)M = M\psi(A) \) for all \( A \in \mathfrak{A} \), where \( \psi \) is an algebra homomorphism. Then \( AKM = K\phi(A)M = KM\psi(A) \), which implies that \( KM \mathcal{H} \) is invariant under \( \mathfrak{A} \). By the minimality of \( K \mathcal{H} \), \( K \mathcal{H} = KM \mathcal{H} \); since \( K \) is injective, we obtain \( M \mathcal{H} = \mathcal{H} \).

(iii) First note that \( AL = L \phi(A) \) implies that either \( L \) is zero or it is injective, because if \( Lx = 0 \) with \( x \neq 0 \), then \( L\phi(A)x = \mathfrak{A}Lx = 0 \). Since \( \phi(\mathfrak{A}) \) is strongly dense in \( \mathfrak{B}(\mathcal{H}) \), this implies \( L \mathcal{H} = 0 \).

Now the equation \( AL = L \phi(A) \) also implies that \( L \mathcal{H} \) is invariant under \( \mathfrak{A} \) and thus there is an operator \( M \) with \( LM = K \). Then

\[
LM\phi(A) = ALM = L\phi(A)M
\]
or, assuming \( L \neq 0 \), \( M\phi(A) = \phi(A)M \). Since \( \phi(\mathfrak{A}) \) is dense in \( \mathfrak{B}(\mathcal{H}) \), \( M \) commutes with \( \phi(\mathfrak{A}) \) and must be a scalar operator.

Before stating the next lemma we fix some notation and terminology. For \( A \in \mathfrak{B}(\mathcal{H}) \) we denote by \( A^{(n)} \) the direct sum of \( n \) copies of \( A \) acting on \( \mathcal{H}^{(n)} \), the direct sum of \( n \) copies of \( \mathcal{H} \); for an algebra \( \mathfrak{A} \) of operators, \( \mathfrak{A}^{(n)} \) will stand for \( \{A^{(n)}: A \in \mathfrak{A}\} \). A **graph subspace** for \( \mathfrak{A}^{(n)} \) is a closed subspace of \( \mathcal{H}^{(n)} \), invariant under \( \mathfrak{A}^{(n)} \), which has the form

\[
\{x \oplus T_1x \oplus \cdots \oplus T_{n-1}x: x \in \mathcal{D}\},
\]
where \( \mathcal{D} \) is a dense linear manifold in \( \mathcal{H} \) and, for each \( i \), \( T_i \) is a linear transformation from \( \mathcal{D} \) into \( \mathcal{H} \). Each \( T_i \) is called a **graph transformation** for \( \mathfrak{A} \). This definition implies, in particular, that \( \mathcal{D} \) and the \( T_i \mathcal{D} \) are invariant under \( \mathfrak{A} \).

The following lemma due to Arveson, will be used; for a proof see [7, p. 143].

**Lemma 2.** Let \( \mathfrak{A} \) be an algebra with the property that every graph transformation for \( \mathfrak{A} \) is a scalar multiple of the identity on \( \mathcal{H} \). Then \( \mathfrak{A} \) is strongly dense in \( \mathfrak{B}(\mathcal{H}) \).

(Note that the hypothesis on \( \mathfrak{A} \) implies that \( \mathfrak{A} \) has no closed invariant subspaces except \( \{0\} \) and \( \mathcal{H} \).)

**Theorem.** Let \( \mathfrak{A} \) be a subalgebra of \( \mathfrak{B}(\mathcal{H}) \). Assume that \( \mathfrak{A} \) has a dense,
invariant operator range contained in every nonzero invariant operator range. Then $\mathfrak{A}$ is strongly dense in $\mathfrak{B}(\mathcal{K})$.

Proof. Let $K \in \mathfrak{B}(\mathcal{K})$ such that $K\mathcal{K}$ is the lower bound of the nonzero invariant operator ranges as hypothesized. The operator $K$ can also be assumed to be injective. Now $AK = K\phi(A)$ for all $A \in \mathfrak{A}$, where $\phi$ is the isomorphism given by Lemma 1.

Let $n$ be any positive integer and let $\mathcal{M}$ be a graph subspace for $\mathfrak{A}^{(n)}$. Then $\mathcal{M}$ can be considered as the graph of a closed linear transformation $T$ from a dense domain $\mathcal{D}$ in $\mathcal{K}$ into $\mathcal{K}^{(n-1)}$ defined by $Tx = T_1x \oplus \cdots \oplus T_{n-1}x$, $x \in \mathcal{D}$. Now $\mathcal{D} = C_1\mathcal{K}$, where $C_1$ is a bounded, injective operator on $\mathcal{K}$ (see, e.g., [4]). Thus

$$\mathcal{M} = \{ C_1y \oplus TC_1y : y \in \mathcal{K} \};$$

since $TC_1$ is closed, it follows from the closed-graph theorem that it is bounded. Hence $T_iC$ is bounded, for each $i$, because $T_iC = P_iTC$, where $P_i$ is a projection. We can now represent $\mathcal{M}$ as

$$\{ C_1x \oplus C_2x \oplus \cdots \oplus C_nx : x \in \mathcal{K} \},$$

where each $C_i$ is a bounded operator and where $C_1$ is, furthermore, injective.

The inclusion $A^{(n)}\mathcal{M} \subseteq \mathcal{M}$ implies that for each $x$ and each $A \in \mathfrak{A}$ there exists a $y$ with $AC_iy = C_iy$ for $i = 1, \ldots, n$. The injectivity of $C_1$ implies that $y$ is unique, and it follows that there is an algebra isomorphism $\psi$ of $\mathfrak{A}$ into $\mathfrak{B}(\mathcal{K})$ such that $AC_i = C_i\psi(A)$ for all $A$ and all $i$. In particular, $C_1\mathcal{K}$ is invariant under $\mathfrak{A}$ and thus $C_1B = K$ for some operator $B$. Thus, by the equation $AK = K\phi(A)$,

$$C_1B\psi(A) = AC_1B = C_1\psi(A)B$$

and, therefore, $B\psi(A) = \psi(A)B$. Now, for each $i$,

$$AC_iB = C_i\psi(A)B = C_iB\psi(A),$$

and Lemma 1 implies that $C_iB = \alpha_iK = \alpha_iC_1B$ for some scalar $\alpha_i$. Thus

$$y \oplus \alpha_2y \oplus \cdots \oplus \alpha_ny \in \mathcal{M},$$

for every $y \in K\mathcal{K}$. Since $K\mathcal{K}$ is dense in $\mathcal{K}$,

$$\mathcal{M} = \{ y \oplus \alpha_2y \oplus \cdots \oplus \alpha_ny : y \in \mathcal{K} \}.$$

We have shown that every $T_i$ is a scalar multiple of the identity on $\mathcal{K}$. The strong density of $\mathfrak{A}$ in $\mathfrak{B}(\mathcal{K})$ now follows from Lemma 2.

**Corollary 1.** If in the lattice of invariant operator ranges for $\mathfrak{A}$ the nonzero elements have a nonzero lower bound, then either $\mathfrak{A}$ is strongly dense in $\mathfrak{B}(\mathcal{K})$ or it has a nontrivial closed, invariant subspace.

**Proof.** If this lower bound represents a dense range, we apply the above theorem; otherwise, the closure of this range is a nontrivial invariant subspace.

**Corollary 2.** If $\mathfrak{A}$ is as in Corollary 1 and if $\mathcal{M}$ is the lower bound for
nonzero, invariant operator ranges, then the restriction $\mathfrak{A}|\mathfrak{M}$ of $\mathfrak{A}$ of the closure of $\mathfrak{M}$ is dense in $B(\mathfrak{M})$.

PROOF. It is easily verified that $\mathfrak{M}$ is the lower bound of the nonzero operator ranges invariant under $\mathfrak{A}|\mathfrak{M}$; thus the theorem is applicable.

QUESTIONS. One natural question poses itself: Can the lattice-theoretic hypothesis of the above results be weakened to require the mere existence of a minimal nonzero element in this lattice? What about the more special case where every nonzero member of the lattice is assumed to contain a minimal nonzero element?

REFERENCES


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