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EXTENSIONS OF Z_2 BORDISM

R. PAUL BEEM

ABSTRACT. The purpose of this note is to characterize the image of any extension homomorphism from unoriented Z_2 bordism to G bordism, where G is a finite abelian group of even order.

1. Introduction. The purpose of this note is to characterize the image of any extension homomorphism from the unoriented bordism of unrestricted involutions, $N_*(Z_2)$, to $N_*(G)$, unoriented G bordism, where G is any finite abelian group of even order. We also have a few remarks concerning more general extension homomorphisms.

Suppose G is a finite abelian group and $t \in G$ has order two. There is an extension homomorphism, which we will denote by e_t from $N_*(Z_2)$ to $N_*(G)$ which is defined on representatives by sending the Z_2 action (M, θ) to $(M \times_{Z_2} G, \theta_t)$, where $\theta_t([m, g], g') = [m, gg']$ and the twisted product is obtained from the Cartesian product by identifying (m, g) with $(\theta(m, -1), tg)$.

Our results are:

THEOREM 1. *If G has two distinct elements of order two, then e_t is the zero homomorphism.*

THEOREM 2. *If G has a unique element t of order two, $t = s^{2^{k-1}}$ for maximum k and*

a. *if $k = 1$, then e_t is a monomorphism of algebras*; or

b. *if $k > 1$, then the image of e_t is a ring with zero products and is isomorphic, as N_* modules, to $N_*(Z_2)/(\text{image } N_*(S^1))$.*

We also note that the image of any extension homomorphism $e: N_*(H) \to N_*(G)$ is an ideal and therefore a subalgebra of $N_*(G)$. In fact, we show that if H is a subgroup of the finite abelian group G, and if $[G : H]$ is even, then although e does not preserve products in general, the image of e is a ring with zero products. If $[G : H]$ is odd, then e is an algebra morphism.

2. Proofs. First consider the case in which G contains at least two distinct elements t and t' of order two. Since G is abelian, these classes generate a copy of $Z_2 \times Z_2$ in G. Hence e_t factors through $N_*(Z_2 \times Z_2)$. But Z_2 extensions are stationary-point free and therefore bound in $N_*(Z_2 \times Z_2)$–see

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[4]—and therefore bound in $N_*(G)$. This concludes the proof of Theorem 1.

One should note that e_t is the zero homomorphism for nonabelian groups G provided that a Sylow 2-subgroup H contains at least two distinct elements of order two. One just notes that the centralizer of H either contains t or another element of order two commuting with t.

LEMMA. *Suppose H is a subgroup of a finite abelian group G and e: $N_*(H) \to N_*(G)$ is the extension homomorphism. If $[G : H]$ is even, then* image *e is a ring with zero products. If $[G : H]$ is odd, then e is a monomorphism of algebras.*

PROOF. Denote the restriction homomorphism from $N_*(G)$ to $N_*(H)$, which remembers only the H action, by r. We know that $re(x) = (G/H)x$, where G/H has trivial H action; see [3]. Hence re is the identity homomorphism if $[G : H]$ is odd and is the zero homomorphism if $[G : H]$ is even.

Next, suppose that $[M, \alpha] = r[M, \theta]$ and that $[N, \beta]$ is an H action. There is the diffeomorphism

$$h: (M \times N) \times_H G \to M \times (N \times_H G)$$

defined by $h[(m, n), g] = (\theta(g, m), [n, g])$ which sends the extension of $\alpha \times \beta$ to $\theta \times$ (extension of β). Hence $e(r(x)y) = xe(y)$. See [2]. Therefore $e(x)e(y) = e(re(x)y)$, which is zero if $[G : H]$ is even and is $e(xy)$ if $[G : H]$ is odd. This concludes the proof of the Lemma.

Note that the equation $e(r(x)y) = xe(y)$ shows that the image of $N_*(H)$ in $N_*(G)$ is an ideal and therefore a subalgebra. It is not true, however, that e always preserves products. For example, let $G = Z_4$ and $H = Z_2$. Suppose that $r(x)$ and $e(y)$ are not zero. According to [1], the kernel of e is the kernel of the derivation d on $N_*(Z_2)$ defined by twisting with the antipodal involution on the circle. Hence $d(r(x)y) = r(x)d(y)$ (since $d \circ r = 0$), which is not zero since $N_*(Z_2)$ is an integral domain. Hence $e(r(x)y) \neq 0 = e(r(x)) \cdot e(y)$.

Note also that the formula $re(x) = (G/H)x$ requires only that H be central in the not necessarily abelian group G. Hence, the Lemma remains true for finite groups G and central subgroups H.

Now suppose that G contains only one element of order two. We write $t = s^{2^{k-1}}$ for maximal k. If $k = 1$, then G/Z_2 has odd order and the Lemma implies that e_t is a monomorphism of algebras. In fact the image of e_t consists of classes of the form $[M \times H, Z_2 \times H]$, where $H \cong G/Z_2$. Note that since a unique element of order two in a group must be central, Theorem 2a is true for all finite groups.

Suppose $k > 1$. Then, according to the Lemma, the image of e_t is a ring with zero products. Since G/Z_{2^k} has odd order, where Z_{2^k} is generated by s, $G \cong Z_{2^k} \times H$, where H has odd order. Therefore the extension from Z_{2^k} to G bordism is a monomorphism of algebras. Hence, it suffices to characterize the image of extension from Z_2 to Z_{2^k} bordism.

Finally, we recall from [1] that the kernel of this extension is identical with the image of the restriction homomorphism from $N_*(S^1)$, the unoriented bordism of circle actions. Therefore, the image of $N_*(Z_2)$ in $N_*(G)$ is isomorphic to $N_*(Z_2)/(\text{image } N_*(S^1))$.

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Department of Mathematics, Indiana University, South Bend, Indiana 46615