DIVISIBILITY PROPERTIES OF THE $q$-TANGENT NUMBERS

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Abstract. The $q$-tangent number $T_{2n+1}(q)$ is shown to be divisible by $(1 + q)(1 + q^2) \cdots (1 + q^n)$. Related divisibility questions are discussed.

1. Introduction. The tangent numbers $T_{2n+1}$ are integers defined by

$$\sum_{n=0}^{\infty} \frac{T_{2n+1} x^n}{(2n+1)!} = \tan x.$$ \hfill (1.1)

Numerous properties of the tangent number are known; in particular [2, p. 259]:

$$T_{2n+1} = 4^{n+1} |G_{2n+2}| / (n + 1),$$ \hfill (1.2)

where $G_n$ is an integer called the Genocchi number. Thus it is clear from (1.2) that $T_{2n+1}$ is always divisible by a high power of 2.

A natural $q$-analog of the tangent numbers is given by

$$\sum_{n=0}^{\infty} \frac{T_{2n+1}(q) x^n}{(q)_{2n+1}} = \frac{\sin_q x}{\cos_q x},$$ \hfill (1.3)

where $(A)_n = (A; q)_n = (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1})$; R. P. Stanley [4] has given a combinatorial interpretation of the polynomials $T_{2n+1}(q)$ which shows that all the coefficients are nonnegative.

One of us [3] has shown that $T_{2n+1}(q)$ is divisible by the cyclotomic polynomials $\phi_2(q), \phi_4(q), \ldots, \phi_{2^n}(q)$ through a study of properties of Gaussian polynomials in cyclotomic fields. Our object here is to derive the following result on $q$-tangent numbers which is analogous to the fact that $T_{2n+1}$ is divisible by a high power of 2:

**Theorem 1.** The polynomial $T_{2n+1}(q)$ is divisible by $(1 + q)(1 + q^2) \cdots (1 + q^n)$.

We conclude with a few comments about other divisibility properties of $T_{2n+1}(q)$ that are derivable using our method. The assertion in Theorem 1 was a conjecture made by M. P. Schützenberger at the combinatorics

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2. Proof of Theorem 1. To prove this result we require two lemmas.

**Lemma 1.** For nonnegative integers \( N \) and \( j \), the expression

\[
\left[ \frac{2N+1}{2j} \right] \frac{(1 + q)(1 + q^2) \cdots (1 + q^j)}{(1 + q^N)(1 + q^{N-1}) \cdots (1 + q^{N-j+1})}
\]

is a polynomial in \( q \), where \( \left[ \frac{N}{M} \right] \) is the Gaussian polynomial

\[
\left[ \frac{N}{M} \right] = \frac{(q)_N}{(q)_M (q)_{N-M}}.
\]

**Proof.** Obviously the expression in question is a rational function and the roots of the denominator are roots of unity. To prove Lemma 1 we need only show that each zero of the denominator appears with at least as large multiplicity in the numerator as in the denominator.

Now if \( \rho \) is a primitive \( k \)th root of unity then \( \rho \) is a simple root of \( 1 - q^M \) if and only if \( k \mid M \). Furthermore we know a priori (due to the recurrences for Gaussian polynomials) that \( \left[ \frac{2N+1}{2j} \right] \) is a polynomial. Consequently for each integer \( l \) with \( 1 \leq l < 2j \), we see that \( l \) must divide at least \( \left[ \frac{2N+1}{2j} \right] \) of the numbers \( 2N + 1, 2N, 2N - 1, \ldots, 2N - 2j + 2 \) (otherwise this Gaussian polynomial would not be a polynomial). Now

\[
\left[ \frac{2N+1}{2j} \right] \frac{(1 + q)(1 + q^2) \cdots (1 + q^j)}{(1 + q^N)(1 + q^{N-1}) \cdots (1 + q^{N-j+1})} = \frac{(1 - q^{2N+1})(1 - q^N)(1 - q^{2N-1})(1 - q^{N-1}) \cdots (1 - q^{2N-2j+3})(1 - q^{N-j+1})}{(1 - q^l)(1 - q^{l-1})(1 - q^{l-2})(1 - q^{l-3}) \cdots (1 - q)(1 - q)}
\]

and one sees that this is the same as the expression for \( \left[ \frac{2N+1}{2j} \right] \) except that each even exponent in numerator and denominator has been divided by 2. Thus the divisibility properties previously described are preserved since the only change is that \( j \) numerator exponents and \( j \) denominator exponents have been divided by 2 which of course does not affect whether a denominator exponent divides a numerator exponent (i.e. if \( l \) is odd and \( l \mid 2M \) then \( l \mid M \), if \( l \) is even and \( l \mid 2M \) then \( \frac{l}{2} \mid M \)). Thus the denominator of

\[
\left[ \frac{2N+1}{2j} \right] \frac{(1 + q)(1 + q^2) \cdots (1 + q^j)}{(1 + q^N)(1 + q^{N-1}) \cdots (1 + q^{N-j+1})}
\]

has no zeros that are not cancelled by those of the numerator. This proves Lemma 1. □

**Lemma 2.** The \( q \)-tangent numbers satisfy
\[ T_{2N+1}(q) + \sum_{j=1}^{N} (-q)^{2j-1} \left[ \frac{2N + 1}{2j} \right] (-1)^j T_{2N+1-2j}(q) \]

(2.4)

\[ = (-1)^N (-q)_{2N}; \]

where \((a)_n = (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), (a)_0 = 1. \]

**Proof.** We have

(2.5) \[ \sum_{n=0}^{\infty} \frac{T_{2n+1}(q)x^{2n+1}}{(q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q)_{2n+1}} / \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q)_n}. \]

Now (here \(i = \sqrt{-1}\))

(2.6)

\[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{i^{n-1}x^n}{(q)_n} \frac{(1 - (-1)^n)}{2} \]

(2.7)

\[ = \frac{1}{2i} \left( \frac{1}{(ix)_\infty} - \frac{1}{(-ix)_\infty} \right) \quad \text{(by [1, p. 19, equation (2.2.5)]);} \]

\[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q)_{2n}} = \sum_{n=0}^{\infty} \frac{i^nx^n}{(q)_n} \frac{(1 + (-1)^n)}{2} \]

Therefore

\[ \sum_{n=0}^{\infty} \frac{T_{2n+1}(q)x^{2n+1}}{(q)_{2n+1}} = \frac{1}{2i} \left( \frac{1}{(ix)_\infty} - \frac{1}{(-ix)_\infty} \right) / \frac{1}{2} \left( \frac{1}{(ix)_\infty} + \frac{1}{(-ix)_\infty} \right) \]

\[ = \frac{1}{i} \frac{(-ix)_\infty - (ix)_\infty}{(-ix)_\infty + (ix)_\infty} = \frac{1}{i} \frac{(-ix)_\infty / (ix)_\infty - 1}{(-ix)_\infty / (ix)_\infty + 1}. \]

Clearing the denominator on the right and utilizing the \(q\)-binomial series

\[ \sum (A)_n z^n = (Az)_\infty \]

[1, p. 17, equation (2.2.1)], we find that

\[ \left( 1 + \sum_{n=0}^{\infty} \frac{(-1)^n (ix)^n}{(q)_n} \right) \sum_{n=0}^{\infty} \frac{T_{2n+1}(q)x^{2n+1}}{(q)_{2n+1}} = \frac{1}{i} \sum_{n=1}^{\infty} \frac{(-1)_n (ix)^n}{(q)_n}. \]

Let us now compare the real parts of the coefficient of \(x^{2N+1}\) in this last identity:

\[ 2T_{2N+1}(q) + \sum_{j=1}^{N} (-1)^j \left[ \frac{2N + 1}{2j} \right] (-1)^j T_{2N+1-2j}(q) = (-1)_{2N+1}(-1)^N, \]

and if we divide each side of this identity by 2 we obtain the result stated in Lemma 2. □
Theorem 1. The polynomial \((1 + q)(1 + q^2) \cdots (1 + q^N)\) divides the polynomial \(T_{2N+1}(q)\).

Proof. The result is immediate for \(N = 0, 1\) since \(T_1 = 1\) and \(T_3 = q(1 + q)\). Let us now assume the result true up to but not including \(N\).

Now
\[
(-q)^{2j-1} \left[ \frac{2N + 1}{2j} \right] = (-q)^j (-q^{j+1})^{j-1} \left[ \frac{2N + 1}{2j} \right] = (1 + q^N)(1 + q^{N-1}) \cdots (1 + q^{N-j+1}) (-q^{j+1})^{j-1}
\]
\[
\times \frac{(1 + q)(1 + q^2) \cdots (1 + q^j)}{(1 + q^N)(1 + q^{N-1}) \cdots (1 + q^{N-j+1})} \left[ \frac{2N + 1}{2j} \right].
\]
Hence by Lemma 1, \((1 + q^N)(1 + q^{N-1}) \cdots (1 - q^{N-j+1})\) is factor of the polynomial \((-q)^{2j-1} \left[ \frac{2N + 1}{2j} \right]\). By the induction hypothesis \((1 + q)(1 + q^2) \cdots (1 + q^{N-j})\) is a factor of \(T_{2N+1-2j}(q)\). Hence for \(1 < j < N\), we see that \((-q)^N\) is a factor of
\[
(-1)^j (-q)^{2j-1} \left[ \frac{2N + 1}{2j} \right] T_{2N+1-2j}(q),
\]
and since \((-q)^N\) is obviously a factor of \((-q)^{2N}\) we deduce from Lemma 2 that \((-q)^N\) is a factor of \(T_{2N+1}(q)\) as well. Thus Theorem 1 follows by induction.

3. Conclusion. First we note that the result mentioned in the Introduction about the divisibility of the \(T_{2n+1}(q)\) by the cyclotomic polynomials \(\Phi_2(q), \Phi_4(q), \ldots, \Phi_{2n}(q)\) now follows from Theorem 1 since \(\Phi_{2n}(q)\) divides \((1 + q^n)\).

We also note that the divisibility of \(T_{2n+1}(q)\) by specific factors of the form \(1 + q^j\) can be handled again by Lemma 2. For example:

Theorem 2. The polynomial \((1 + q)^n\) is a factor of the \(q\)-tangent number \(T_{2n+1}(q)\).

Proof. The result is obvious for \(n = 0, 1\) since \(T_1(q) = 1\) and \(T_3(q) = q(1 + q)\). Assume the theorem true up to but not including \(n\). Now since \(1 + q^{2M+1} = (1 + q)(1 - q + q^2 - \cdots + q^{2M})\), we see that \((1 + q)^j\) is a factor of \((-q)^{2j-1}\). By the induction hypothesis \((1 + q)^{N-j}\) is a factor of \(T_{2N+1-2j}(q)\). Hence \((1 + q)^N\) is a factor of
\[
(-q)^{2j-1} \left[ \frac{2N + 1}{2j} \right] (-1)^j T_{2N+1-2j}(q),
\]
and since \((1 + q)^N\) is also a factor of \((-q)^{2N}\), we deduce from Lemma 2 that \((1 + q)^N\) is a factor of \(T_{2N+1}(q)\).

References

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