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SHORTER NOTES

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THERE ARE NO UNIQUELY HOMOGENEOUS SPACES

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ABSTRACT. One could say a continuum is uniquely homogeneous if for each pair of points there is a unique homeomorphism taking the one point to the other. Ungar showed that such spaces are topological groups with no automorphisms. This note shows there are no such nontrivial groups.

Introduction. In his paper, *On all kinds of homogeneous spaces*, Ungar proves the following:

If G is either a compact metric space or a locally compact, locally connected, separable metric space which is uniquely homogeneous, then G is an abelian topological group, and the group of topological automorphisms of G is trivial.

He showed that such a finite dimensional G would be a Lie group and thus have too many homeomorphisms to be uniquely homogeneous. Using topological arguments together with some structure theorems for topological groups, it is possible to show there are no uniquely homogeneous continua at all. This completely answers a question to this effect posed by Burgess at the 1955 Wisconsin topology conference. Alternatively one uses the following result about topological groups.

RESULT. Let G be a T_2 locally compact topological group with $\mathrm{Aut}(G) = \{\mathrm{id}\}$. Then $G = \{0\}$ or Z_2.

PROOF. All inner automorphisms of G are trivial so G is abelian; and $x \to -x$ is trivial so all elements have order 2. By Zorn's lemma it is easy to show that an automorphism of any open subgroup extends to all of G. Since G is torsion and locally compact it contains a compact open subgroup H. By the extension result, $\mathrm{Aut}(H)$ must be trivial. Let X be the character group of H. X is discrete and $\mathrm{Aut}(X)$ is trivial (anti-isomorphic with $\mathrm{Aut}(H)$). Now X must be $\{0\}$ or Z_2 otherwise it contains an open subgroup isomorphic to $Z_2 \times Z_2$ which has nontrivial automorphisms. So H is Z_2 or $\{0\}$, G must be discrete, so $G = \{0\}$ or Z_2.

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COMMENT. Since $G = \{0\}$ or Z_2 towers of open subgroups are pretty trivial. Is the axiom of choice really necessary?

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