A SHORT PROOF OF THE DAWKINS-HALPERIN THEOREM

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Abstract. A brief proof is presented, of the Dawkins-Halperin Theorem, that if $D$ is a finite dimensional division algebra with centre $F$, then the direct limits of appropriately-sized matrix rings over $D$ and $F$ are isomorphic; the isomorphism can be given in a form suitable for comparing cohomology groups of $D$ and $F$.

For a ring $R$, we denote the ring of $t$ by $t$ matrices with entries from $R$, by $M_t R$. There is an obvious map from $R$ to $M_t R$:

$$
\Delta(t): R \to M_t R, \quad r \mapsto \begin{bmatrix}
    r & & \\
    & \ddots & \\
    & & r
\end{bmatrix}.
$$

Let $D$ denote a division ring of dimension $n^2$ over its centre $F$. The main result of [1], asserts that if $t = n^2$, then as $F$-algebras

$$
\lim_{i \to \infty} M_i D \cong \lim_{i \to \infty} M_i F,
$$

the maps in both limits being $\Delta(t)$. However, the proof there is exceptionally obscure and complicated. We give a short natural proof, requiring only the Noether-Skolem Theorem:

[3, Theorem 4.3.1]. Let $C$ be a finite dimensional simple $F$-algebra with centre $F$, and let $A, B$ be simple subalgebras of $C$, each with centre $F$. Then any $F$-algebra isomorphism from $A$ to $B$ can be extended to an inner automorphism of $C$.

Theorem. Let $D$ be a division algebra of dimension $n^2$ over its centre, the field $F$. Set $t = n^2$, and for each positive integer $i$, let $j_i$ denote the map $M_i R \subset M_i D$ induced by the inclusion of $F$ in $D$. Form the $F$-algebras $\lim_{i \to \infty} M_i F, \lim_{i \to \infty} M_i D$, with $\Delta(t)$ as the maps in the limits.

There exist inner automorphisms $\psi_i, \varphi_i$ of $M_i D, M_i F$ respectively, so that if $\alpha_i: M_i F \to M_i D$ are defined by $\alpha_i = \psi_i^{-1} j_i \varphi_i$, then the $\alpha_i$ are compatible with the maps in the direct limits, and the induced map

$$
\lim_{i \to \infty} \alpha_i: \lim_{i \to \infty} M_i F \to \lim_{i \to \infty} M_i D
$$

is an $F$-algebra isomorphism.
Let $k : D \to M_i F$ be a fixed $F$-algebra homomorphism (for instance, the right regular representation of $D$), and define maps $k_i : M_i D \to M_{i+1} F$ to be the maps on the matrix rings induced by $k$. We may form the limit, $S$, of the diagram (1):

$$
\begin{array}{c}
F \leftarrow D \\
\downarrow k \\
M_i F \to M_i D \to M_{i+1} F \\
\downarrow \ \\
M_{i+2} F \\
\end{array}
$$

Then $S$ is algebra isomorphic to $\lim_{i \to \infty} M_i F$.

**Proof.** Pick a fixed map $k$, such as the right regular representation, and separate (1) into two rows, rows 2 and 3 of diagram (2).

$$
\begin{array}{c}
\cdots \to M_i F \\
\downarrow k_i \\
M_{i+2} F \\
\end{array} \to \begin{array}{c}
\cdots \\
\downarrow k_{i+1} \\
M_{i+2} F \\
\end{array}
$$

Because $M_i F$ and $M_i D$ are cofinal in diagram (1), $\lim_j$ is actually an isomorphism (with inverse, $\lim k_i$) from the limit of row 2 to the limit of row 3. We shall construct inner $\psi_i : M_i F \to M_i F$ (row 1 to row 2) and $\psi_i : M_i D \to M_i D$ (row 4 to row 3) so that the whole of (2) commutes.

Define $\psi_0 : D \to D$ to be the identity map. Assuming $\psi_s$ have been defined for $0 < s < i$, so that rows 3 and 4 commute, we see

$$
j_{i+1} k_i \psi_i(M_i D) \simeq \Delta(M_i D)
$$

as $F$-subalgebras of $M_{i+1} D$, the isomorphism obtained by pulling back the image of $\Delta$, and applying $j_{i+1} k_i \psi_i$. By the Noether-Skolem Theorem, there exists an invertible $V$ in $M_{i+1} D$ so that this isomorphism is implemented by conjugation with $V$. Define $\psi_{i+1}(A) = V A V^{-1}$; then $\psi_{i+1} \Delta = j_{i+1} k_i \psi_i$, concluding the induction.

Thus $\psi_i$ defines a map between the limits of the fourth and third rows; it follows that $\lim \psi_i^{-1}$ (from row 3 to row 4) exists and is the inverse. In particular, $\lim \psi_i^{-1}$ is an algebra isomorphism.

The same process allows us to construct a similar isomorphism, $\lim \psi_i$, from the limit of row 1 to the limit of row 2, with each $\psi_i$ inner. Since $\lim j_i$ is an isomorphism, and

$$
\lim (\psi_i^{-1} j_i \psi_i) = (\lim \psi_i^{-1})(\lim j_i)(\lim \psi_i),
$$

setting $\alpha_i = \psi_i^{-1} j_i \psi_i$ (mapping down the columns of (2)), we see that $\lim \alpha_i$ is an isomorphism, and the final statement is an immediate consequence. \qed

The form of the isomorphism obtained above is particularly useful in computing the homology or cohomology of $D$ relative to that of $F$ (see [2], for
an application), because such functors usually commute with direct limits, and change inner automorphisms into the identity.

Theorem 2 of [1] effectively asserts that if a division ring $D$ can be represented as a limit, $\lim D_i$, with each $D_i$ a finite dimensional central division algebra over $F$, then

$$\lim_{\Delta(m)} M_n F \cong \lim_{\Delta(m)} M_n D,$$

where $m$ varies over all products of numbers of the form $[D_i: F]$, and $n$ varies similarly. (Of course, $\Delta(m): M_n F \to M_n F$ is defined only if $r = mn$.) One can easily prove that this follows from our theorem above.

REFERENCES


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