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GLOBAL CHARACTERIZATIONS OF THE SPHERE

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ABSTRACT. Let S be an ovaloid in Euclidean three-space E^3 with Gaussian curvature $K > 0$ and let K_{II} be the curvature of the second fundamental form II of S. We give some global characterizations of the sphere by the curvature K_{II} which generalize some results of R. Schneider [4], D. Koutroufiotis [2] and the well-known "H-Satz" theorem of H. Liebmann.

Let S be a closed and sufficiently smooth surface in Euclidean three-space E^3 with Gaussian curvature $K > 0$, i.e. an *ovaloid*. By appropriate orientation the second fundamental form II defines a nondegenerate Riemannian metric on S. Let H be the mean curvature of S and K_{II} be the curvature of II. Then the equality

$$K_{\mathrm{II}} = H + Q - (8K^2)^{-1}\nabla_{\mathrm{II}}K \tag{1}$$

holds identically on S, where Q is a certain nonnegative function and ∇_{II} denotes the first Beltrami operator (square of the gradient) with respect to II [4]. On the closed surface S there exist points P_0, P_1 so that

$$K(P_0) = \max_{p \in S} K(P), \quad K(P_1) = \min_{p \in S} K(P).$$

Thus we have $\nabla_{\mathrm{II}}K(P_0) = \nabla_{\mathrm{II}}K(P_1) = 0$, and since $Q \geqslant 0$, (1) yields $K_{\mathrm{II}}(P_0) \geqslant H(P_0)$ and $K_{\mathrm{II}}(P_1) \geqslant H(P_1)$.

THEOREM 1. *Let S be an ovaloid in E^3. If there exists a function $f(x, y)$ which satisfies the following conditions*:

(a) *$f(x, y)$ is increasing in x and decreasing in y,*

(b) *the function $F(x, y) := f(x, y)/x$ $(x \neq 0)$ is decreasing in x, and if $K_{\mathrm{II}} = f(H, K)$ identically on S, then S is a sphere.*

PROOF. Since $K_{\mathrm{II}} = f(H, K)$ and $H \geqslant \sqrt{K}$,

$$\frac{f(H(P), K(P))}{\sqrt{K(P)}} \geqslant \frac{f\left(\sqrt{K(P)}\,, K(P)\right)}{\sqrt{K(P)}} \geqslant \frac{f\left(\sqrt{K(P_0)}\,, K(P)\right)}{\sqrt{K(P_0)}}$$
$$\geqslant \frac{f(H(P_0), K(P))}{H(P_0)} \geqslant \frac{f(H(P_0), K(P_0))}{H(P_0)} \geqslant 1$$

for every $P \in S$. Then we have

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$$K_{\mathrm{II}}(P) = f(H(P), K(P)) \geqslant \sqrt{K(P)}$$

and S is a sphere (cf. [**5**, Corollary 5, p. 383]).

THEOREM 1′. *Let S be an ovaloid in E^3. If there exists a function $f(x, y)$ which satisfies the following conditions*:

(a) *$f(x, y)$ is increasing in x,*

(b) *the function $F(x, y) := f(x, y)/x$ $(x \neq 0)$ is decreasing in x,*

(c) *the function $G(x, y) := f(x, y)/\sqrt{y}$ $(y > 0)$ is increasing in y, and if $K_{\mathrm{II}} = f(H, K)$ identically on S, then S is a sphere.*

PROOF. We have

$$\frac{f(H(P), K(P))}{\sqrt{K(P)}} \geqslant \frac{f\left(\sqrt{K(P)}\,, K(P)\right)}{\sqrt{K(P)}} \geqslant \frac{f\left(\sqrt{K(P_1)}\,, K(P)\right)}{\sqrt{K(P)}}$$
$$\geqslant \frac{f\left(\sqrt{K(P_1)}\,, K(P_1)\right)}{\sqrt{K(P_1)}} \geqslant \frac{f(H(P_1), K(P_1))}{H(P_1)} \geqslant 1$$

for every $P \in S$. Hence,

$$K_{\mathrm{II}}(P) = f(H(P), K(P)) \geqslant \sqrt{K(P)}$$

and S is a sphere [**5**, Corollary 5, p. 383].

For appropriate choices of the constants r and s, the function $f(x, y) = cx^s y^r$, with $c > 0$ constant, satisfies the hypotheses of Theorems 1 or 1′. For example, if $K_{\mathrm{II}} = cH^sK^r$ on S with $0 \leqslant s \leqslant 1$ and $r \leqslant 0$ or $r \geqslant \frac{1}{2}$, then S is a sphere. This result generalizes those of Schneider [**4**, Theorem 1] (for surfaces in E^3) and Koutroufiotis [**2**, Theorems 1 and 2].

LEMMA 1. *Let $\bar{P}$ be a point on an ovaloid S where H attains its maximum. Then $K_{\mathrm{II}}(\bar{P}) \geqslant H(\bar{P})$.*

PROOF. According to [**1**, p. 7] the identity

$$2H(K_{\mathrm{II}} - H)(H^2 - K) = \tfrac{1}{2}K\nabla_{\mathrm{II}}(H, H^2/K) - \frac{1}{4}\nabla(H^2/K, K)$$

is valid, which easily implies

$$\begin{aligned} &2H(K_{\mathrm{II}} - H)(H^2 - K) \\ &\quad = \tfrac{1}{2}K\nabla_{\mathrm{II}}\left(H, \frac{H^2}{K}\right) - \frac{H}{2K}\nabla(H, K) + \frac{H^2}{4K^2}\nabla K. \end{aligned} \tag{2}$$

Here $\nabla(\varphi, \psi)$ (resp. $\nabla_{\mathrm{II}}(\varphi, \psi)$) denotes the inner product of the gradients of φ and ψ with respect to I (resp. II) and $\nabla\varphi := \nabla(\varphi, \varphi)$. The right-hand side of (2) is nonnegative at $\bar{P}$, since

$$\nabla_{\mathrm{II}}\left(H(\bar{P}), H^2(\bar{P})/K(\bar{P})\right) = \nabla\left(H(\bar{P}), K(\bar{P})\right) = 0 \quad \text{and} \quad \nabla K(\bar{P}) \geqslant 0.$$

Since $H^2(\bar{P}) \geqslant K(\bar{P})$, the left side of (2) yields $K_{\mathrm{II}}(\bar{P}) \geqslant H(\bar{P})$.

The following results are similar to Simon's [**5**, Lemma 6 and Theorem, p. 383].

LEMMA 2. *Let S be an ovaloid in E^3. If there exists $\bar{P} \in S$ where K_{II} takes its minimum and H takes its maximum, then S is a sphere.*

PROOF. We get, for every $P \in S$,

$$K_{II}(P) \geqslant K_{II}(\bar{P}) \geqslant H(\bar{P}) \geqslant H(P) \geqslant \sqrt{K(P)}$$

and the assertion follows [**5**, Corollary 5, p. 383].

Finally, using Lemma 2, we can prove in the same way as in [**5**, Theorem, p. 383]:

THEOREM 2. *Let S be an ovaloid in E^3. If there exists a function $\Phi(x, y)$ which is increasing (resp. decreasing) in both variables and strictly monotonic in at least one of its variables and if $\Phi(H(P), K_{II}(P)) = 0$ for all $P \in S$, then S is a sphere.*

This theorem is a common generalization of Schneider's [**4**, Theorem 1] (for surfaces in E^3) and the well-known "H-Satz" theorem of Liebmann (cf. [**3**, p. 163]).

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