GLOBAL CHARACTERIZATIONS OF THE SPHERE

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Abstract. Let $S$ be an ovaloid in Euclidean three-space $E^3$ with Gaussian curvature $K > 0$ and let $K_{II}$ be the curvature of the second fundamental form $II$ of $S$. We give some global characterizations of the sphere by the curvature $K_{II}$ which generalize some results of R. Schneider [4], D. Koutroufiotis [2] and the well-known “H-Satz” theorem of H. Liebmann.

Let $S$ be a closed and sufficiently smooth surface in Euclidean three-space $E^3$ with Gaussian curvature $K > 0$, i.e. an ovaloid. By appropriate orientation the second fundamental form $II$ defines a nondegenerate Riemannian metric on $S$. Let $H$ be the mean curvature of $S$ and $K_{II}$ be the curvature of $II$. Then the equality

$$K_{II} = H + Q - (8K^2)^{-1} V_{II} K$$

holds identically on $S$, where $Q$ is a certain nonnegative function and $V_{II}$ denotes the first Beltrami operator (square of the gradient) with respect to $II$ [4]. On the closed surface $S$ there exist points $P_0, P_1$ so that

$$K(P_0) = \max_{P \in S} K(P), \quad K(P_1) = \min_{P \in S} K(P).$$

Thus we have $V_{II} K(P_0) = V_{II} K(P_1) = 0$, and since $Q > 0$, (1) yields $K_{II}(P_0) > H(P_0)$ and $K_{II}(P_1) > H(P_1)$.

Theorem 1. Let $S$ be an ovaloid in $E^3$. If there exists a function $f(x, y)$ which satisfies the following conditions:

(a) $f(x, y)$ is increasing in $x$ and decreasing in $y$,

(b) the function $F(x, y) := f(x, y)/x \ (x \neq 0)$ is decreasing in $x$, and if $K_{II} = f(H, K)$ identically on $S$, then $S$ is a sphere.

Proof. Since $K_{II} = f(H, K)$ and $H > \sqrt{K}$,

$$\frac{f(H(P), K(P))}{\sqrt{K(P)}} > \frac{f(\sqrt{K(P)}, K(P))}{\sqrt{K(P)}} > \frac{f(\sqrt{K(P_0)}, K(P))}{\sqrt{K(P_0)}}$$

for every $P \in S$. Then we have

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\[ K_{\Pi}(P) = f(H(P), K(P)) > \sqrt{K(P)} \]
and \( S \) is a sphere (cf. [5, Corollary 5, p. 383]).

**Theorem 1'.** Let \( S \) be an ovaloid in \( E^3 \). If there exists a function \( f(x, y) \) which satisfies the following conditions:

(a) \( f(x, y) \) is increasing in \( x \),
(b) the function \( F(x, y) := f(x, y)/x \) \((x \neq 0)\) is decreasing in \( x \),
(c) the function \( G(x, y) := f(x, y)/\sqrt{y} \) \((y > 0)\) is increasing in \( y \), and if \( K_{\Pi} = f(H, K) \) identically on \( S \), then \( S \) is a sphere.

**Proof.** We have

\[
\frac{f(H(P), K(P))}{\sqrt{K(P)}} > \frac{f(\sqrt{K(P)}, K(P))}{\sqrt{K(P)}} > \frac{f(\sqrt{K(P_1)}, K(P_1))}{\sqrt{K(P_1)}} > \frac{f(H(P_1), K(P_1))}{H(P_1)} > 1
\]
for every \( P \in S \). Hence,

\[ K_{\Pi}(P) = f(H(P), K(P)) > \sqrt{K(P)} \]
and \( S \) is a sphere [5, Corollary 5, p. 383].

For appropriate choices of the constants \( r \) and \( s \), the function \( f(x, y) = cx^s y^r \), with \( c > 0 \) constant, satisfies the hypotheses of Theorems 1 or 1'. For example, if \( K_{\Pi} = c H^s K^r \) on \( S \) with \( 0 < s < 1 \) and \( r < 0 \) or \( r > 1/2 \), then \( S \) is a sphere. This result generalizes those of Schneider [4, Theorem 1] (for surfaces in \( E^3 \)) and Koutroufiotis [2, Theorems 1 and 2].

**Lemma 1.** Let \( \bar{P} \) be a point on an ovaloid \( S \) where \( H \) attains its maximum. Then \( K_{\Pi}(\bar{P}) > H(\bar{P}) \).

**Proof.** According to [1, p. 7] the identity

\[ 2H(K_{\Pi} - H)(H^2 - K) = \frac{1}{2} K\nabla_{\Pi}(H, H^2/K) - \frac{1}{4} \nabla(H^2/K, K) \]
is valid, which easily implies

\[ 2H(K_{\Pi} - H)(H^2 - K) = \frac{1}{2} K\nabla_{\Pi}(H, \frac{H^2}{K}) - \frac{H}{2K} \nabla(H, K) + \frac{H^2}{4K^2} \nabla K. \]

Here \( \nabla(\varphi, \psi) \) (resp. \( \nabla_{\Pi}(\varphi, \psi) \)) denotes the inner product of the gradients of \( \varphi \) and \( \psi \) with respect to \( I \) (resp. \( \Pi \)) and \( \nabla \varphi := \nabla(\varphi, \varphi) \). The right-hand side of (2) is nonnegative at \( \bar{P} \), since

\[ \nabla_{\Pi}(H(\bar{P}), H^2(\bar{P})/K(\bar{P})) = \nabla(H(\bar{P}), K(\bar{P})) = 0 \quad \text{and} \quad \nabla K(\bar{P}) > 0. \]

Since \( H^2(\bar{P}) > K(\bar{P}) \), the left side of (2) yields \( K_{\Pi}(\bar{P}) > H(\bar{P}) \).
The following results are similar to Simon’s [5, Lemma 6 and Theorem, p. 383].

**Lemma 2.** Let $S$ be an ovaloid in $E^3$. If there exists $\bar{P} \in S$ where $K_{II}$ takes its minimum and $H$ takes its maximum, then $S$ is a sphere.

**Proof.** We get, for every $P \in S$,

$$K_{II}(P) > K_{II}(\bar{P}) > H(\bar{P}) > H(P) > \sqrt{K(P)}$$

and the assertion follows [5, Corollary 5, p. 383].

Finally, using Lemma 2, we can prove in the same way as in [5, Theorem, p. 383]:

**Theorem 2.** Let $S$ be an ovaloid in $E^3$. If there exists a function $>(x,y)$ which is increasing (resp. decreasing) in both variables and strictly monotonic in at least one of its variables and if $\Phi(H(P), K_{II}(P)) = 0$ for all $P \in S$, then $S$ is a sphere.

This theorem is a common generalization of Schneider’s [4, Theorem 1] (for surfaces in $E^3$) and the well-known “$H$-Satz” theorem of Liebmann (cf. [3, p. 163]).

**References**


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