GLOBAL CHARACTERIZATIONS OF THE SPHERE

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Abstract. Let $S$ be an ovaloid in Euclidean three-space $E^3$ with Gaussian curvature $K > 0$ and let $K_{II}$ be the curvature of the second fundamental form $II$ of $S$. We give some global characterizations of the sphere by the curvature $K_{II}$ which generalize some results of R. Schneider [4], D. Koutroufiotis [2] and the well-known "H-Satz" theorem of H. Liebmann.

Let $S$ be a closed and sufficiently smooth surface in Euclidean three-space $E^3$ with Gaussian curvature $K > 0$, i.e. an ovaloid. By appropriate orientation the second fundamental form $II$ defines a nondegenerate Riemannian metric on $S$. Let $H$ be the mean curvature of $S$ and $K_{II}$ be the curvature of $II$. Then the equality

$$K_{II} = H + Q - (8K^2)^{-1} \nabla_{II}K$$

holds identically on $S$, where $Q$ is a certain nonnegative function and $\nabla_{II}$ denotes the first Beltrami operator (square of the gradient) with respect to $II$ [4]. On the closed surface $S$ there exist points $P_0$, $P_1$ so that

$$K(P_0) = \max_{P \in S} K(P), \quad K(P_1) = \min_{P \in S} K(P).$$

Thus we have $\nabla_{II}K(P_0) = \nabla_{II}K(P_1) = 0$, and since $Q > 0$, (1) yields $K_{II}(P_0) > H(P_0)$ and $K_{II}(P_1) > H(P_1)$.

**Theorem 1.** Let $S$ be an ovaloid in $E^3$. If there exists a function $f(x, y)$ which satisfies the following conditions:

(a) $f(x, y)$ is increasing in $x$ and decreasing in $y$,

(b) the function $F(x, y) := f(x, y)/x$ ($x \neq 0$) is decreasing in $x$, and if $K_{II} = f(H, K)$ identically on $S$, then $S$ is a sphere.

**Proof.** Since $K_{II} = f(H, K)$ and $H > \sqrt{K}$,

$$\frac{f(H(P), K(P))}{\sqrt{K(P)}} \geq \frac{f(\sqrt{K(P)}, K(P))}{\sqrt{K(P)}} \geq \frac{f(\sqrt{K(P_0)}, K(P))}{\sqrt{K(P_0)}} \geq \frac{f(H(P_0), K(P))}{H(P_0)} \geq \frac{f(H(P_0), K(P_0))}{H(P_0)} \geq 1$$

for every $P \in S$. Then we have
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\[ K_{II}(P) = f(H(P), K(P)) \geq \sqrt{K(P)} \]

and \(S\) is a sphere (cf. [5, Corollary 5, p. 383]).

**THEOREM 1'.** Let \(S\) be an ovaloid in \(E^3\). If there exists a function \(f(x, y)\) which satisfies the following conditions:

(a) \(f(x, y)\) is increasing in \(x\),

(b) the function \(F(x, y) := f(x, y)/x\) \((x \neq 0)\) is decreasing in \(x\),

(c) the function \(G(x, y) := f(x, y)/\sqrt{y}\) \((y > 0)\) is increasing in \(y\), and if \(K_{xx} = f(H, K)\) identically on \(S\), then \(S\) is a sphere.

**PROOF.** We have

\[
\frac{f(H(P), K(P))}{\sqrt{K(P)}} \geq \frac{f(\sqrt{K(P)}, K(P))}{\sqrt{K(P)}} > \frac{f(\sqrt{K(P_1)}, K(P_1))}{\sqrt{K(P_1)}} \geq \frac{f(H(P_1), K(P_1))}{H(P_1)} \geq 1
\]

for every \(P \in S\). Hence,

\[ K_{II}(P) = f(H(P), K(P)) \geq \sqrt{K(P)} \]

and \(S\) is a sphere [5, Corollary 5, p. 383].

For appropriate choices of the constants \(r\) and \(s\), the function \(f(x, y) = cx^sy^\gamma\), with \(c > 0\) constant, satisfies the hypotheses of Theorems 1 or 1'. For example, if \(K_{xx} = ch^sK^r\) on \(S\) with \(0 < s < 1\) and \(r < 0\) or \(r = \frac{1}{2}\), then \(S\) is a sphere. This result generalizes those of Schneider [4, Theorem 1] (for surfaces in \(E^3\)) and Koutroufiotis [2, Theorems 1 and 2].

**LEMMA 1.** Let \(\bar{P}\) be a point on an ovaloid \(S\) where \(H\) attains its maximum. Then \(K_{II}(\bar{P}) \geq H(\bar{P})\).

**PROOF.** According to [1, p. 7] the identity

\[ 2H(K_{II} - H)(H^2 - K) = \frac{1}{2} K\nabla_{II}(H, H^2/K) - \frac{1}{4} \nabla(H^2/K, K) \]

is valid, which easily implies

\[ 2H(K_{II} - H)(H^2 - K) = \frac{1}{2} K\nabla_{II}(H, \frac{H^2}{K}) - \frac{H}{2K} \nabla(H, K) + \frac{H^2}{4K^2} \nabla K. \]

Here \(\nabla(\varphi, \psi)\) (resp. \(\nabla_{II}(\varphi, \psi)\)) denotes the inner product of the gradients of \(\varphi\) and \(\psi\) with respect to \(I\) (resp. \(II\)) and \(\nabla \varphi := \nabla(\varphi, \varphi)\). The right-hand side of (2) is nonnegative at \(\bar{P}\), since

\[ \nabla_{II}(H(\bar{P}), H^2(\bar{P})/K(\bar{P})) = \nabla(H(\bar{P}), K(\bar{P})) = 0 \quad \text{and} \quad \nabla K(\bar{P}) > 0. \]

Since \(H^2(\bar{P}) \geq K(\bar{P})\), the left side of (2) yields \(K_{II}(\bar{P}) \geq H(\bar{P})\).
The following results are similar to Simon’s [5, Lemma 6 and Theorem, p. 383].

**Lemma 2.** Let $S$ be an ovaloid in $E^3$. If there exists $P \in S$ where $K_{II}$ takes its minimum and $H$ takes its maximum, then $S$ is a sphere.

**Proof.** We get, for every $P \in S$,

$$K_{II}(P) \geq K_{II}(\bar{P}) \geq H(\bar{P}) \geq H(P) > \sqrt{K}(P)$$

and the assertion follows [5, Corollary 5, p. 383].

Finally, using Lemma 2, we can prove in the same way as in [5, Theorem, p. 383]:

**Theorem 2.** Let $S$ be an ovaloid in $E^3$. If there exists a function $\Phi(x, y)$ which is increasing (resp. decreasing) in both variables and strictly monotonic in at least one of its variables and if $\Phi(H(P), K_{II}(P)) = 0$ for all $P \in S$, then $S$ is a sphere.

This theorem is a common generalization of Schneider’s [4, Theorem 1] (for surfaces in $E^3$) and the well-known “$H$-Satz” theorem of Liebmann (cf. [3, p. 163]).

**References**