ON A THEOREM OF E. LUKACS

PAUL EMBRECHTS

ABSTRACT. We prove that an integral transform of measures on a locally compact abelian group, which satisfies both the uniqueness and the convolution property, is closely related to the Fourier-Stieltjes transform. This extends a result obtained by Lukacs for the real line.

1. Introduction. Given a certain kernel $K$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{C}$, consider the following integral transform $f$ of an arbitrary probability distribution function $F$ on $\mathbb{R}$:

$$
f(s) = \int_{-\infty}^{+\infty} K(s, x) \, dF(x).
$$

If we take $K$ such that this transform exists for all $F$ and moreover satisfies the uniqueness property (U) and the convolution property (C), then Lukacs [1] proved that:

$$
K(s, t) = e^{\iota A(s)}
$$

where $A$ is a real valued function with a dense range. Hence (U) and (C) force $f$ to be 'nearly' the Fourier-Stieltjes transform of $F$.

Using the same method as Lukacs [1], we extend this result to integral transforms of finite, complex valued measures on a locally compact abelian (L.C.A.) group.

2. Main theorem. Let $G$ be a L.C.A. group and $\Gamma$ its dual group. Then $\Gamma$ is the set of all continuous complex functions $\gamma$ on $G$ for which $|\gamma(x)| = 1$ for all $x \in G$ and for which $\gamma(x + y) = \gamma(x) \cdot \gamma(y)$ for all $x, y \in G$. Such $\gamma$ is called a character of $G$.

Let us also denote by $\mathcal{M}(G)$ the set of all complex valued, regular Borel measures $\mu$ on $G$ for which the total variation norm $\|\mu\|$ is finite.

The convolution of two measures $\mu, \nu$ will be denoted by $\mu \ast \nu$. Finally, we write $\hat{\mu}$ for the Fourier-Stieltjes transform of $\mu$ i.e. for all $\gamma \in \Gamma$:

$$
\hat{\mu}(\gamma) = \int_{G} \gamma(x) \, d\mu(x).
$$

Now we can state the generalization of Lukacs' result:

**Theorem.** Let $K(x, \gamma)$, as a complex valued mapping defined on $G \times \Gamma$, be
continuous and bounded in \( x \in G \) for all \( \gamma \in \Gamma \); suppose that for all \( \mu \in \mathcal{M}(G) \) the following transform exists:

\[
\tilde{\mu}(\gamma) = \int_G K(x, \gamma) \, d\mu(x), \quad \gamma \in \Gamma;
\]

then necessary and sufficient conditions for \( K \) to satisfy:

- **(U):** for all \( \gamma \in \Gamma \): \( \tilde{\mu}(\gamma) = \tilde{v}(\gamma) \iff \mu = v \),
- **(C):** for all \( \gamma \in \Gamma \): \( (\mu \ast v)(\gamma) = \tilde{\mu}(\gamma) \cdot \tilde{v}(\gamma) \)

are that there exists a function \( g \) from \( \Gamma \) to \( \Gamma \) such that

\[
K(x, \gamma) = g(\gamma)(x)
\]

and \( g \) has a dense range.

Moreover this \( g \) is unique.

**Proof.** Sufficiency is simple. Indeed:

\[
\tilde{\mu}(\gamma) = \int_G g(\gamma)(x) \, d\mu(x) = \tilde{\mu}(g(\gamma)).
\]

Using the fact that \( g(\Gamma) \) is dense and the continuity of \( \tilde{\mu} \) we conclude \( (U) \) and \( (C) \) from the corresponding properties of the Fourier-Stieltjes transform. (See Rudin [4, pp. 15–17].)

Suppose now that \( (U) \) and \( (C) \) hold. Let us denote by \( \delta(a) \) the Dirac measure at \( a \in G \). It is easy to verify that:

\[
(1) \quad \delta(a) \ast \delta(b) = \delta(a + b), \quad \text{for all } a, b \in G.
\]

Applying \( (C) \) to \( \delta(a) \ast \delta(b) \) and using \( (1) \) we get:

\[
K(a + b, \gamma) = \delta(a + b)(\gamma) = (\delta(a) \ast \delta(b))(\gamma)
\]

\[
= \delta(a)(\gamma) \cdot \delta(b)(\gamma) = K(a, \gamma) \cdot K(b, \gamma), \quad \text{for all } a, b \in G.
\]

Moreover, putting \( b = 0 \) in \( (2) \) we get:

\[
K(a, \gamma) = K(a, \gamma) \cdot K(0, \gamma), \quad \text{for all } \gamma \in \Gamma \text{ and } a \in G.
\]

Hence

\[
K(0, \gamma) = 1, \quad \text{for all } \gamma \in \Gamma.
\]

But then, for all \( x \in G \), \( |K(x, \gamma)| = 1 \), because if not there would exist a \( x_0 \in G \) with \( |K(x_0, \gamma)| = r > 1 \). (If \( r < 1 \), use \( -x_0 \) and \( K(x_0, \gamma) \cdot K(-x_0, \gamma) = 1 \).)

Hence \( |K(nx_0, \gamma)| = r^n \) for all \( n \in \mathbb{N} \), contradicting the boundedness of \( K \).

The continuity of \( K \) now implies that for all \( \gamma \in \Gamma \) there exists a \( \gamma' \in \Gamma \) such that

\[
K(x, \gamma) = \gamma'(x) \quad \text{for all } x \in G.
\]

By the Pontryagin Duality Theorem (Rudin [4, p. 27]) this \( \gamma' \) is unique. Hence the function \( g \) from \( \Gamma \) to \( \Gamma \), defined by \( g(\gamma) = \gamma' \), is well defined. So

\[
K(x, \gamma) = g(\gamma)(x).
\]

Hence \( \tilde{\mu}(\gamma) = \tilde{\mu}(g(\gamma)) \); so using \( (U) \):
for all $\gamma \in \Gamma$, \( \tilde{\mu} (\gamma) = \tilde{\nu} (\gamma) \iff \mu = \nu \iff \tilde{\mu} (g(\gamma)) = \tilde{\nu} (g(\gamma)) \).

The last equivalence together with (U) for Fourier-Stieltjes transforms implies that \( g(\Gamma) \) is dense in \( \Gamma \) which proves the theorem.

By way of example, take \( G = \mathbb{R}^{n} \) and \( \mu \in \mathcal{M}_{b} (\mathbb{R}^{n}) \); then the theorem holds for

\[
\tilde{\mu} (s) = \int_{\mathbb{R}^{n}} e^{isg(t)} \, d\mu (t)
\]

where \( s \in \mathbb{R}^{n} \) and \( g \) has a dense range.

This follows from the theorem, the fact that the dual group of \( \mathbb{R}^{n} \) equals \( \mathbb{R}^{n} \) and its characters are of the form \( \exp (its) \). (See Rudin \cite[p. 12]{4}.)

3. Remarks. (a) An analogous theorem for a more general transform under continuity conditions is given in Lukacs \cite{2}, where a similar generalization can be given. This, however, was already noted by the referee in Lukacs \cite[p. 6]{2}.

(b) Lukacs' theorem follows as a special case by taking \( G = \mathbb{R} \) and \( \mathcal{M}_{b} (\mathbb{R}) \) replaced by the set of distribution functions on \( \mathbb{R} \). As shown, his proof applies without change to the L.C.A. case. However, the proof of the additional statement that \( A \) should separate points (see Lukacs \cite[p. 510]{2}) is false; in a later version of the result this fact was omitted. (See Lukacs \cite[p. 100]{3}.)

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REFERENCES


DEPARTEMENT WISKUNDE, KATHOLIEKE UNIVERSITEIT TE LEUVEN, CELESTIJNENLAAN 200B, 3030 HEVERLEE, BELGIUM