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ON A THEOREM OF E. LUKACS

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ABSTRACT. We prove that an integral transform of measures on a locally compact abelian group, which satisfies both the uniqueness and the convolution property, is closely related to the Fourier-Stieltjes transform. This extends a result obtained by Lukacs for the real line.

1. Introduction. Given a certain kernel K from $\mathbf{R} \times \mathbf{R}$ to $\mathbf{C}$, consider the following integral transform f of an arbitrary probability distribution function F on $\mathbf{R}$:

$$f(s) = \int_{-\infty}^{+\infty} K(s, x)\, dF(x).$$

If we take K such that this transform exists for all F and moreover satisfies the uniqueness property (U) and the convolution property (C), then Lukacs **[1]** proved that:

$$K(s, t) = e^{itA(s)}$$

where A is a real valued function with a dense range. Hence (U) and (C) force f to be 'nearly' the Fourier-Stieltjes transform of F.

Using the same method as Lukacs **[1]**, we extend this result to integral transforms of finite, complex valued measures on a locally compact abelian (L.C.A.) group.

2. Main theorem. Let G be a L.C.A. group and Γ its dual group. Then Γ is the set of all continuous complex functions γ on G for which $|\gamma(x)| = 1$ for all $x \in G$ and for which $\gamma(x + y) = \gamma(x) \cdot \gamma(y)$ for all $x, y \in G$. Such γ is called a character of G.

Let us also denote by $\mathfrak{M}(G)$ the set of all complex valued, regular Borel measures μ on G for which the total variation norm $\|\mu\|$ is finite.

The convolution of two measures μ, ν will be denoted by $\mu * \nu$. Finally, we write $\hat{\mu}$ for the Fourier-Stieltjes transform of μ i.e. for all $\gamma \in \Gamma$:

$$\hat{\mu}(\gamma) = \int_G \gamma(x)\, d\mu(x).$$

Now we can state the generalization of Lukacs' result:

THEOREM. *Let $K(x, \gamma)$, as a complex valued mapping defined on $G \times \Gamma$, be*

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continuous and bounded in $x \in G$ *for all* $\gamma \in \Gamma$; *suppose that for all* $\mu \in \mathfrak{M}(G)$ *the following transform exists*:

$$\tilde{\mu}(\gamma) = \int_G K(x, \gamma)\, d\mu(x), \qquad \gamma \in \Gamma;$$

then necessary and sufficient conditions for K to satisfy:

(U): *for all* $\gamma \in \Gamma$: $\tilde{\mu}(\gamma) = \tilde{\nu}(\gamma) \Leftrightarrow \mu = \nu$,

(C): *for all* $\gamma \in \Gamma$: $(\mu * \nu)^{\sim}(\gamma) = \tilde{\mu}(\gamma) \cdot \tilde{\nu}(\gamma)$

are that there exists a function g from Γ *to* Γ *such that*

$$K(x, \gamma) = g(\gamma)(x)$$

and g has a dense range.

Moreover this g is unique.

Proof. Sufficiency is simple. Indeed:

$$\tilde{\mu}(\gamma) = \int_G g(\gamma)(x)\, d\mu(x) = \hat{\mu}(g(\gamma)).$$

Using the fact that $g(\Gamma)$ is dense and the continuity of $\hat{\mu}$ we conclude (U) and (C) from the corresponding properties of the Fourier-Stieltjes transform. (See Rudin [**4**, pp. 15–17].)

Suppose now that (U) and (C) hold. Let us denote by $\delta(a)$ the Dirac measure at $a \in G$. It is easy to verify that:

(1) $$\delta(a) * \delta(b) = \delta(a + b), \quad \text{for all } a, b \in G.$$

Applying (C) to $\delta(a) * \delta(b)$ and using (1) we get:

(2) $$\begin{aligned} K(a + b, \gamma) &= \tilde{\delta}(a + b)(\gamma) = (\delta(a) * \delta(b))^{\sim}(\gamma) \\ &= \tilde{\delta}(a)(\gamma) \cdot \tilde{\delta}(b)(\gamma) = K(a, \gamma) \cdot K(b, \gamma), \quad \text{for all } a, b \in G. \end{aligned}$$

Moreover, putting $b = 0$ in (2) we get:

$$K(a, \gamma) = K(a, \gamma) \cdot K(0, \gamma), \quad \text{for all } \gamma \in \Gamma \text{ and } a \in G.$$

Hence

$$K(0, \gamma) = 1, \quad \text{for all } \gamma \in \Gamma.$$

But then, for all $x \in G$, $|K(x, \gamma)| = 1$, because if not there would exist a $x_0 \in G$ with $|K(x_0, \gamma)| = r > 1$. (If $r < 1$, use $-x_0$ and $K(x_0, \gamma) \cdot K(-x_0, \gamma) = 1$.)

Hence $|K(nx_0, \gamma)| = r^n$ for all $n \in \mathbf{N}$, contradicting the boundedness of K. The continuity of K now implies that for all $\gamma \in \Gamma$ there exists a $\gamma' \in \Gamma$ such that

$$K(x, \gamma) = \gamma'(x) \quad \text{for all } x \in G.$$

By the Pontryagin Duality Theorem (Rudin [**4**, p. 27]) this γ' is unique. Hence the function g from Γ to Γ, defined by $g(\gamma) = \gamma'$, is well defined. So

$$K(x, \gamma) = g(\gamma)(x).$$

Hence $\tilde{\mu}(\gamma) = \hat{\mu}(g(\gamma))$; so using (U):

$$\text{for all } \gamma \in \Gamma, \quad \tilde{\mu}(\gamma) = \tilde{\nu}(\gamma) \Leftrightarrow \mu = \nu \Leftrightarrow \hat{\mu}(g(\gamma)) = \hat{\nu}(g(\gamma)).$$

The last equivalence together with (U) for Fourier-Stieltjes transforms implies that $g(\Gamma)$ is dense in Γ which proves the theorem.

By way of example, take $G = \mathbf{R}^n$ and $\mu \in \mathfrak{M}(\mathbf{R}^n)$; then the theorem holds for

$$\tilde{\mu}(s) = \int_{\mathbf{R}^n} e^{itg(s)}\, d\mu(t)$$

where $s \in \mathbf{R}^n$ and g has a dense range.

This follows from the theorem, the fact that the dual group of $\mathbf{R}^n$ equals $\mathbf{R}^n$ and its characters are of the form $\exp(its)$. (See Rudin [**4**, p. 12].)

3. Remarks. (a) An analogous theorem for a more general transform under continuity conditions is given in Lukacs [**2**], where a similar generalization can be given. This, however, was already noted by the referee in Lukacs [**2**, p. 6].

(b) Lukacs' theorem follows as a special case by taking $G = \mathbf{R}$ and $\mathfrak{M}(\mathbf{R})$ replaced by the set of distribution functions on $\mathbf{R}$. As shown, his proof applies without change to the L.C.A. case. However, the proof of the additional statement that A should separate points (see Lukacs [**2**, p. 510]) is false; in a later version of the result this fact was omitted. (See Lukacs [**3**, p. 100].)

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