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QUASI-AFFINE SURFACES WITH G_a-ACTIONS

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ABSTRACT. A normal quasi-affine surface over an algebraically closed field of characteristic zero which has a G_a-action is shown to have a geometric quotient if and only if the action is without fixed points. If the surface is factorial affine, and the action without fixed points, the surface is the product of a curve and G_a.

A quasi-affine variety is an open subset of an affine variety.

Let X be a normal quasi-affine variety of dimension two over the algebraically closed field k of characteristic zero on which the additive algebraic group G_a acts nontrivially (i.e., such that there exist x in X and t in G_a with $tx \neq x$). For a geometric quotient X/G_a [**3**, p. 724] to exist it is necessary that G_a have no fixed points on X [**1**, 6.4(b), p. 175]. We show here that this necessary condition is also sufficient (Theorem 2). It is known that the condition is insufficient if X has dimension three [**3**, Example 1, p. 727]. We show further that if X is affine and factorial then there is a curve C in X such that $G_a \times C \to X$ by $(t, c) \to tc$ is a G_a-equivariant isomorphism (Theorem 3). We recall that the action of G_a on X is locally trivial if $X \to X/G_a$ is a locally trivial principal G_a-bundle over X/G_a and that the action is proper if $G_a \times X \to X \times X$ by $(t, x) \to (tx, x)$ is a proper morphism. It is known that a proper action is locally trivial [**3**, Proposition 6, p. 725], that a locally trivial action on a factorial affine is proper [**3**, Theorem 7, p. 726] and that a proper action has a separated geometric quotient [**3**, Theorem 4, p. 725]. We give examples to show that a locally trivial action on a nonsingular affine surface need not be proper or have a separated quotient (Example 1) and that an action on a nonsingular affine surface may have an affine quotient but need not be proper, or even locally trivial (Example 2). These examples also yield a nonnormal affine threefold Y with G_a-action such that Y/G_a exists and is affine and such that there is a closed subvariety V of Y such that $V \to Y/G_a$ is an isomorphism, but the bijection $G_a \times V \to Y$ by $(t, v) \to tv$ is not an isomorphism.

NOTATIONS AND CONVENTIONS. Throughout, k denotes an algebraically closed field of characteristic zero.

A prevariety X over k is a reduced irreducible algebraic k-scheme, which we identify with its set of closed points with the Zariski topology. We let $k[X]$

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denote $\Gamma(X, O_X)$. A surface is a prevariety of dimension two. A variety is a separated prevariety.

Let X be a prevariety with G_a-action and let f be in $k[X]$. There exist $f_0, \ldots, f_n$ in $k[X]$ such that $f(tx) = \sum_{i=0}^n f_i(x)t^i$ for all x in X and t in G_a. If $f_n \neq 0$, let $\operatorname{ord}(f) = n$. If f, g are in $k[X]$ then $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$, and $\operatorname{ord}(f) = 0$ if and only if f is invariant. Since $0x = x$ for all x in X, $f_0 = f$. Since $f((x + t)x) = f(s(tx))$ for x, t in G_a and x in X, it follows that $f_j(tx) = \sum_{i=j}^n \binom{i}{j} f_i(x)t^{i-j}$, so $\operatorname{ord}(f_j) = n - j$.

LEMMA 1. *Let X be a quasi-affine variety with nontrivial G_a-action. Then there exists a nonzero invariant f in $k[X]$ such that the quotient of $U = X - f^{-1}(0)$ by G_a exists and is quasi-affine.*

PROOF. Since the action is nontrivial, there is a g in $k[X]$ with $\operatorname{ord}(g) = n > 0$. Then $\operatorname{ord}(g_{n-1}) = 1$ and $(g_{n-1})_1 = ng_n$ is a nonzero invariant. Let $f = ng_n$, and let $U = X - f^{-1}(0)$. Then $h = g_{n-1}/f$ is in $k[U] = k[X][1/f]$, $\operatorname{ord}(h) = 1$ and $h_1 = 1$. Let $W = h^{-1}(0)$. As in [3, Lemma 5, p. 725] it follows that $G_a \times W \to U$ by $(t, w) \to tw$ is a G_a-equivariant isomorphism with inverse $u \to (h(u), -h(u)u)$. Thus U/G_a exists and equals W. It is clear that W is quasi-affine.

THEOREM 2. *Let W be a normal quasi-affine surface on which G_a acts without fixed points. Then the geometric quotient W/G_a exists.*

PROOF. Let $B = k[W]$ and let f be as in Lemma 1. By [6, Lemma 7, p. 220], $k(W)^{G_a}$ is the quotient field of B^{G_a}. Since $k(W)^{G_a}$ is finitely generated over k there exist $f_1, \ldots, f_n$ in B^{G_a} such that $A = k[f_1, \ldots, f_n]$ is normal, and the quotient field of A is $k(W)^{G_a}$. There is an affine variety X in $k^{(n)}$ such that $k[X] = A$ and such that ϕ: $W \to X$ by $\phi(w) \to (f_1(w), \ldots, f_n(w))$ is a morphism. It is clear that X is a normal, hence nonsingular, curve. Since $\phi(W)$ is irreducible, its closure is either X or a point. But if $\phi(W)$ is a point, all f_i are constant so X is a point. Thus ϕ is dominant. Let $w \in W$. Then $\phi^{-1}\phi(w)$ is G_a-stable in W, closed, and not all of W. Thus $\phi^{-1}\phi(w)$ is a finite (disjoint) union of orbits. By [7, Theorem 2, p. 221] these orbits are closed, so every irreducible component of $\phi^{-1}\phi(w)$ is an orbit. Since G_a has no fixed points on W, these orbits are one-dimensional, so every irreducible component of $\phi^{-1}\phi(w)$ has dimension $1 = \dim(W) - \dim(X)$. By [1, Corollary, p. 81] if N is an open neighborhood of w, $\phi(N)$ is an open neighborhood of $\phi(w)$. Thus $\phi(W) = X_0$ is open in X, and ϕ: $W \to X_0$ is an open surjection to the nonsingular curve X_0. Let $U = W - f^{-1}(0)$. By Lemma 1, U/G_a exists, and $U \to X_0$ factors as $U \to U/G_a \to X_0$. The fibres of ϕ are finite unions of orbits so $U/G_a \to X_0$ has finite fibres. Since $k[X_0]$ has quotient field $k(W)^{G_a} = k(U)^{G_a} = k(U/G_a)$, this map is also birational. Since U/G_a and X_0 are normal, Zariski's main theorem implies that the map is an open immersion. Let $U_0 = \phi(U)$. Now $X_0 - U_0$ is finite, say $X_0 - U_0 = \{x_1, \ldots, x_r\}$. We can write $\phi^{-1}(x_i) = C_{i,1} \cup \cdots \cup C_{i,n(i)}$ where the $C_{i,j}$ are disjoint orbits. Call an integer valued function α on $\{1, \ldots, k\}$ a selection function if for each i, $1 \leqslant \alpha(i) \leqslant n(i)$. For each selection function α let $W_\alpha = U \cup C_{1,\alpha(1)} \cup \cdots \cup C_{r,\alpha(r)}$. Then W_α is open in W and ϕ: $W_\alpha \to X_0$ is a surjective, open, separable orbit map, and hence a quotient map [**1**, 6.2, p. 173]. It is clear that W is the union of W_α's over all selection functions α, so W locally has a quotient and hence W/G_a exists globally.

We will see below that W/G_a need not be separated, even in the case that W is an affine surface. For factorial affine surfaces, a much stronger result then Theorem 2 holds:

THEOREM 3. *Let W be a factorial affine surface with a fixed point free G_a-action. Then W/G_a is affine and there is a G_a-equivariant isomorphism $G_a \times W/G_a \to W$.*

PROOF. As is well known, $k[W]^{G_a}$ is a factorial ring. Choose f in $k[W]$ such that $\operatorname{ord}(f) = 1$ and f_1 has a minimal number of prime factors (count all prime factors, not just distinct ones). There must be an irreducible factor g of f with $\operatorname{ord}(g) = 1$. Then $f = ag$ where $\operatorname{ord}(a) = 0$, so $f_1 = (ag)_1 = a(g_1)$. By choice of f, a has no prime factors, so a is a unit and f is irreducible. We note that this argument shows that if $\operatorname{ord}(h) = 1$ and $h_1 = f_1$ then h is irreducible. Let C be a nonempty component of $f_1^{-1}(0)$. Then C must be a G_a-orbit, and $f|C$ is invariant, so $f|C$ is constant, say $f|C = \lambda$. Let $h = f - \lambda$. Then $\operatorname{ord}(h) = 1$ and $h_1 = f_1$ so h is irreducible, and $h|C = 0$ so $C \subset h^{-1}(0)$. Since W is factorial, $h^{-1}(0)$ is irreducible and $C = h^{-1}(0)$. It follows that the ideal $k[W]h$ is G_a-stable and hence contains a nonzero invariant divisible by h. Since $\operatorname{ord}(h) = 1$, this is a contradiction. Thus $f_1^{-1}(0)$ is empty and f_1 is a unit. Replace f by f/f_1. Then $\operatorname{ord}(f) = 1$ and $f_1 = 1$. Let $W_0 = f^{-1}(0)$. As in [**3**, Lemma 5, p. 724] it follows that $G_a \times W_0 \to W$ by $(t, x) \to tx$ is an isomorphism, and $W/G_a = W_0$ is affine.

The following examples illustrate the limitations of the above results, as well as those of [**3**]. The examples are all related, and as we introduce notations we will retain them. Some computations will be only outlined, or even omitted.

We begin by considering the action of G_a on $k^{(3)}$ given by $\gamma \cdot (x_1, x_2, x_3) = (x_1, x_2 + \gamma x_1, x_3 + (2x_2 + 1)\gamma + x_1\gamma^2)$. Let W in $k^{(3)}$ be the zeros of $\phi = x_1x_3 - x_2^2 - x_2$. It is clear that ϕ is irreducible so that W is an affine surface. The partials of ϕ have $(0, -\frac{1}{2}, 0)$ as their only common zero, which is outside W, so W is nonsingular. Also, W is G_a-stable. We use a, b, c for the coordinates on W which are the restrictions of x_1, x_2, x_3 respectively.

W meets the plane $x_1 = \lambda$ for $\lambda \neq 0$ in the parabola $x_3 = \lambda^{-1}(x_2^2 + x_2)$ and the plane $x_1 = 0$ in the pair of lines l_1: $x_1 = x_2 = 0$ and l_2: $x_1 = 0, x_2 = -1$. (We show below that these curves are orbits.) Also, W contains the lines L_1: $x_2 = x_3 = 0$ and L_2: $x_2 = -1, x_3 = 0$.

If (a, b, c) and (a, b', c') are in W and $a \neq 0$, then $a^{-1}(b' - b) \cdot (a, b, c) = (a, b', c')$ so the two points are in the same orbit. Also for any c, c' in k, $(c - c') \cdot (0, -1, c) = (0, -1, c')$ and $(c' - c) \cdot (0, 0, c) = (0, 0, c')$. Thus l_1

and l_2 are also orbits. The fixed points of G_a on $k^{(3)}$ are the line $x_1 = 0$, $x_2 = -\frac{1}{2}$ which does not meet W, so G_a acts without fixed points on W. Let $U_1 = W - l_2$ and $U_2 = W - l_1$. The morphism f_1: $G_a \times L_1 \to U_1$ given by $f_1(t, x) = tx$ is a G_a-equivariant bijection: we have unique orbits if $a \neq 0$ and $f_1(t, (0, 0, 0)) = (0, 0, t)$ covers the orbit l_1. Since U_1 is normal f_1 is an isomorphism. Similarly, f_2: $G_a \times L_2 \to U_2$ by $f_2(t, x) = tx$ is a G_a-equivariant isomorphism. Thus the G_a-action on W is locally trivial.

To describe the quotient, we let L be the prevariety obtained by identifying L_1 and L_2 where a is not zero: $(a, 0, 0)$ is identified with $(a, -1, 0)$ for $a \neq 0$. Define q: $W \to L$ so that $q | U_i$ is $pr_2 f_i^{-1}$ followed by the inclusion of L_i in L. Then q: $W \to L$ is a geometric quotient for W by G_a. Summarizing:

EXAMPLE 1. There is a nonsingular affine surface with G_a-action such that the action is locally trivial but the quotient is not separated, and hence the action is not proper.

By Theorem 3, W cannot be factorial. For example, a, b, c, and $b + 1$ are irreducible in $k[W]$, but $ac = b(b + 1)$. In fact, the divisor class group of W is Z: for q: $W \to L$ is a fibration with fibre $k^{(1)}$, and the exact sequence [**5**, p. 72] shows that $\text{Pic}(W) = \text{Pic}(L)$. It is clear that $\text{Pic}(L) = Z$ (L is a line with the origin doubled and so L has one nontrivial divisor class coming from the extra origin) and since W is nonsingular its Picard group and class group are equal.

We next consider the morphism σ: $W \to W$ given by $\sigma(a, b, c) = (-a, -b - 1, -c)$. Since $\sigma^2 = 1$, σ is an automorphism of W. If (a, b, c) were a fixed point of σ, we would have $a = -a$, $b = -b - 1$, and $c = -c$, so $(a, b, c) = (0, -\frac{1}{2}, 0)$ which is not in W. Thus σ is fixed point free. Let $\Sigma = \{1, \sigma\}$ be the group of automorphisms generated by σ. Since W is affine, the quotient $W' = W/\Sigma$ exists. Since σ has no fixed points, $k[W]$ is a finitely generated projective $k[W']$-module by [**2**, 1.3(f), p. 4]. By [**6**, 21.E, p. 156], $k[W']$ is regular. Thus W' is a nonsingular affine surface. Let $[a, b, c]$ denote the image of (a, b, c) in W'. A computation shows that the actions of G_a and Σ commute on W, so $\gamma \cdot [a, b, c] = [\gamma \cdot (a, b, c)]$ defines an action of G_a on W'. Since a^2 is a G_a and Σ invariant function on W, we have a morphism p: $W' \to k^{(1)}$ given by $p[a, b, c] = a^2$.

We show that p gives a geometric quotient of W' by G_a: by [**1**, 6.6, p. 179] it suffices to show that p is an orbit map. Thus suppose $p[a, b, c] = p[a', b', c']$, so $a^2 = (a')^2$. If $a = 0$, then $a' = 0$, so $(a, b, c) = (0, -1, c)$ or $(0, 0, c)$. Since $\sigma(0, -1, c) = (0, 0, c)$ we may assume $b = 0$, and similarly that $b' = 0$. But then $(c' - c) \cdot [0, 0, c] = [0, 0, c']$ so $[a, b, c]$ and $[a', b', c']$ are in the same orbit. If $a \neq 0$, then $a' = a$ or $a' = -a$. If $a' = -a$, $\sigma(a', b', c') = (a, b'', c'')$ so we may assume $a = a'$. Then $a^{-1}(b' - b) \cdot [a, b, c] = [a', b', c']$ so again both points are in the same orbit. Also $p(\gamma \cdot [a, b, c]) = a^2 = p[a, b, c]$, so p is an orbit map. Thus W' has a quotient, and $W'/G_a = k^{(1)}$ is affine. If the G_a-action on W' were locally trivial, W' would be a G_a-bundle over the affine variety $k^{(1)}$, and hence W' would be globally trivial; i.e., there would be a closed subvariety F of W' on a G_a equivariant isomorphism $G_a \times F \to W'$. By [3, Lemma 5, p. 725], this implies the existence of a function f in $k[W']$ such that $f(\gamma x) = f(x) + \gamma$ for all x in W' and γ in G_a. But $k[W']$ is contained in $k[W]$ so this implies by [3, Lemma 5, p. 725] again that W is globally trivial. This, in turn, means that W has an affine quotient, but L is not affine. Summarizing:

EXAMPLE 2. There is a nonsingular affine surface with G_a-action which has an affine quotient, but the action is not locally trivial.

By [3, Proposition 6, p. 725], the action of G_a on W' is not proper. By Theorem 3, W' is not factorial. For example, we have $k[W'] = k[W]^{\Sigma}$, so a^2, c^2 and ac belong to $k[W]$ and are irreducible, but $a^2c^2 = (ac)^2$. In fact, W' has class group $Z/2Z$. One can show that $k[W'] = k[a^2, c^2, ac, a(2b + 1), c(2b + 1)]$. Then $k[W'][a^{-2}] = k[a^2, a^{-2}, a^{-1}(2b + 1)]$ is factorial, so by [4, 7.2, p. 36] the class group of W' is generated by the components of the zeros of a^2. The zeros of a^2 in W are the lines l_1 and l_2 which are conjugate under σ, so the zeros of a^2 have a unique component in W', and hence this component defines a torsion divisor class on W'. Since $\operatorname{Pic}(W) = Z$, this divisor class is in the kernel of $\operatorname{Pic}(W') \to \operatorname{Pic}(W)$, which by [4, 16.1, p. 82] is isomorphic to $H^1(\Sigma, k[W]^*)$. There is a ring homomorphism $k[W] \to k[x, t]$ (polynomials) which sends a to x, b to xt, and c to $t + xt^2$. The image $k[x, xt, t + xt^2]$ has dimension two, so the homomorphism is an injection and $k[W]^* = k^*$, so $H^1(\Sigma, k[W]^*) = Z/2Z$.

Finally, we consider the subvariety X of $W' \times W'$ consisting of all pairs (x, y) such that $p(x) = p(y)$. X is closed in $W' \times W'$ and pr_2: $X \to W'$ is a surjection. G_a acts on X by $\gamma \cdot (x, y) = (\gamma x, y)$, and h: $G_a \times W' \to X$ by $h(t, x) = (tx, x)$ is a G_a-equivariant bijection since p is an orbit map. Moreover, pr_2: $X \to W'$ is an orbit map, so by [1, 6.6, p. 179] pr_2 is a geometric quotient of X by G_a. Let $F = h(0 \times W')$. Then F is closed in X and pr_2: $F \to W'$ is an isomorphism. But h is not an isomorphism: if it were, then $G_a \times W' \to W' \times W'$ by $(t, x) \to (tx, x)$ would be a closed immersion, hence proper, so the action of G_a on W' would be proper, contrary to Example 2. Summarizing:

EXAMPLE 3. There is an irreducible affine threefold X with G_a-action which has a nonsingular affine quotient, and there is a closed subset F of X such that $F \to X/G_a$ is an isomorphism, but $G_a \times F \to X$ is not an isomorphism.

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