ON GENERICITY AND COMPLEMENTS OF MEASURE ZERO SETS IN FUNCTION SPACES

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ABSTRACT. Generic properties of function spaces have been of particular interest in dynamical systems and singularity theory. The underlying assumption has been that the complement of a dense $G_d$ set is sparse enough to be considered unlikely. Nevertheless, in infinite dimensional spaces, even dense $G_d$'s may have measure zero. Since there is no one canonical measure on an infinite dimensional Frechet space, notions of measure zero have not often been considered. Here we use a notion of Haar measure zero on abelian Polish groups due to Christensen [1]. We show that those sections of a finite dimensional vector bundle over a compact manifold whose jets are transverse to a submanifold of the jet bundle are complements of sets of Haar measure zero.

An abelian Polish group $G$ is an abelian topological group with group operation $+$ such that $G$ has a separable Hausdorff topology and such that there exists at least one complete metric on $d$ inducing the given topology.

A universally measurable subset of $G$ is a subset of $G$ that is measurable for every probability measure defined on the Borel sets of $G$.

We will say that a universally measurable subset $A$ of $G$ has Haar measure zero if there is a (nonunique) probability measure $du$ on $G$, called a testing measure, such that for any $g \in G$, $\int_G \chi_{A + g} \, du = 0$. Here $A + g$ is the $g$-translate of $A$, and for any $S \subset G$, $\chi_S$ is the characteristic function of $S$. That is, the set $A$ has Haar measure zero if $A$ and all its translates have measure zero with respect to the testing measure.

We observe without proof that if $G$ were a locally compact group with Haar measure $dh$, then the notion we have defined is equivalent to saying that $\int_G \chi_A \, dh = 0$. We also observe that the countable union of Haar measure zero sets is a set of Haar measure zero. The justification for interest in this notion of measure zero is that it seems to be a suitable one for doing calculus on abelian Polish groups. In particular, there is a generalization of a theorem of Rademacher saying that Lipschitz mappings between certain abelian Polish groups have directional derivatives linear in the direction a.e. in this sense.

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See [1] for details and further references.

Let \( M \) be a compact, connected \( C^\infty \) manifold. Let \( B \) be a vector bundle over \( M \) with finite dimensional fiber. We will use the symbol \( C^\infty(B) \) to denote the set of \( C^\infty \) sections of \( B \) with addition of sections as the group operation. For any nonnegative integer \( k \), we will let \( J^k(B) \) denote the \( k \)-jet bundle of sections from \( M \) to \( B \). There is a well-known transversality theorem that says that if \( W \) is a submanifold of \( J^k(B) \), then there is a residual subset \( R \) of \( G \) such that if \( g \in R \) then \( j^k g : M \to J^k(B) \) is transversal to \( W \) (see [2, II. 4.9]). Here \( j^k g \) is the \( k \)-jet mapping of \( g \). Our main result will be

I. THEOREM. Let \( G = C^\infty(B) \). Let \( W \) be a submanifold of \( J^k(B) \) and let \( R \) be the subset of those sections \( g \in G \) such that \( j^k(g) : M \to J^k(B) \) is transverse to \( W \). Then \( G \) is an abelian Polish group and \( R \) is the complement of a set of Haar measure zero in \( G \).

To prove the theorem, we will need some lemmas.

II. LEMMA. \( G \) is separable.

PROOF OF LEMMA II. Since \( M \) is compact, we can find a positive integer \( p \) and for each \( i = 1, 2, \ldots, p \) we can find a \( C^\infty \) function \( \phi_i : M \to [0, 1] \) such that

1. (i) \( \phi_i \) has compact support \( C_i \),
   (ii) \( C_i \subset U_i \subset M \), where \( U_i \) is open and diffeomorphic to an open subset of \( \mathbb{R}^m \), and
   (iii) \( \sum_{i=1}^{p} \phi_i(x) = 1 \) for any \( x \in M \).

As a result, any section \( g \in G \) can be written as a finite sum \( g = \sum \phi_i g \). Here \( \phi_i g \) is a \( C^\infty \) section of \( B \) with compact support in \( C_i \).

We will show the existence of a countable set of sections that are dense in the set \( G_i = \{ \phi_i g : g \in G \} \). This will complete the proof of Lemma II.

We may, without loss of generality, assume that the restriction of \( B \) to \( U_i \) is trivial. That is, \( B|_{U_i} \approx U_i \times \mathbb{R}^n \) for some integer \( n \). In particular, we may identify \( U_i \) with an open subset of \( \mathbb{R}^m \) and identify sections of \( B|_{U_i} \) with \( C^\infty \) maps from \( U_i \) to \( \mathbb{R}^n \). By the Weierstrass approximation theorem, see [3], those maps whose component functions are polynomials with rational coefficients can be used to uniformly approximate any other map up to order \( k \) in the \( C^k \) sup norm on \( C_i \). In particular, if we denote those countably many polynomial maps by \( \{ p_j : j \in \mathbb{N} \} \), then \( \{ \phi_i p_j : j \in \mathbb{N} \} \) is dense in \( G_i \). This proves the lemma.

II. LEMMA. There is a complete metric \( d \) on \( G \) that induces the Whitney \( C^\infty \) topology.

PROOF. The Whitney \( C^\infty \) topology is the coarsest topology which is finer than the Whitney \( C^k \) topology for any finite \( k \). If \( G^k = C^k(B) \) is the set of \( C^k \) sections of \( B \) in the Whitney \( C^k \) topology, then there is a complete metric \( d^k \) on \( G^k \) (see [2, p. 43]). Since \( G \subset G^k \), there is an induced metric on \( G \) which we will also call \( d^k \). If \( g, g' \in G \), we will define a metric on \( G \) by
It is not hard to see that \( d \) is complete and induces the proper topology on \( G \).

Now let us prove Theorem 1. As in [2, II. 4.9], we choose a countable open cover of \( W \) by open sets \( W_1, W_2, \ldots \) such that:

1. (i) the closure of \( W_i \) in \( J^k(B) \) is contained in \( W \);
   (ii) \( \overline{W_i} \) is compact;
   (iii) there is a chart \( U_i \) on \( M \) such that \( B|_{U_i} \approx U_i \times \mathbb{R}^n \) for some integer \( n \) and such that \( \pi(\overline{W_i}) \subset U_i \times \mathbb{R}^n \), where \( \pi: J^k(B) \to B \) is the natural projection mapping; and
   (iv) \( U_i \) is compact.

If we let \( T_i = \{ g \in G : \text{for } \forall x \in \overline{W_i}, g \text{ is transversal to } W \} \), then just as in [2], each \( T_i \) is open and dense and \( R = \bigcap T_i \). Let us show that for each \( i \), the set \( CT_i \), the complement of \( T_i \), is a set of Haar measure zero.

Let \( \pi_i \) be the projection of \( J^k(B) \) onto \( M \). Note that \( W_i \subset \pi_i^{-1}(U_i) \). We can always assume that we have chosen the \( W_i \) and \( U_i \) small enough so that there exist a finite integer \( q \) and \( q \) sections \( g_1, g_2, \ldots, g_q \in G \) such that:

1. (i) for each \( g \in G \) we can define a map \( \tilde{g}: \mathbb{R}^q \to G \) by \( \tilde{g}(s_1, \ldots, s_q)(x) = (s_1g_1 + \cdots + s_qg_q + g)(x) \);
   (ii) the map \( \tilde{g} \) from \( \mathbb{R}^q \times M \to J^k(B) \), defined by \( \tilde{g}((s_1, \ldots, s_q), x) = j^k(\tilde{g}(s_1, \ldots, s_q))(x) \), is onto \( \pi_i^{-1}(U_i) \);
   (iii) for each fixed \( x \in U_i \), the map \( \tilde{g}((\cdot \cdot \cdot ), x) \) from \( \mathbb{R}^q \) to \( J^k(B) \) is surjective at each point in its domain.

To see that such \( g_1, \ldots, g_q \) can exist, we identify \( U_i \) with a neighborhood of zero in \( \mathbb{R}^m \) and let \( g_1, \ldots, g_q \) be the set of all distinct maps to \( \mathbb{R}^n \) with only one nonzero entry that is a monomial of order \( \leq k \) in \( m \) variables.

Let \( S = \hat{0}(\mathbb{R}^q) \) be the image of \( \mathbb{R}^q \) in \( G \). Here \( 0 \) is the zero section of \( B \). We will induce a probability measure on \( S \) by using a probability measure on \( \mathbb{R}^q \) which assigns probability zero to sets of Lebesgue measure zero. If \( R \subset S \), we define the measure of \( R \) to be the measure of \( (0)^{-1}(R) \). If \( R \subset G \), but \( R \) is not a subset of \( S \), we define the measure of \( R \) to be the measure of \( R \cap S \).

We wish to show that \( CT_i \) has measure zero with respect to the testing measure we have just defined.

By hypotheses 3(i)–(iii), for any \( g \in G \), the map \( g: \mathbb{R}^q \times M \to J^k(B) \) is transversal to \( W \) on \( U_i \). By Lemma II.4.6 of [2], the set \( A_g \) of points \( (s_1, \ldots, s_q) \in \mathbb{R}^q \) such that \( g((s_1, \ldots, s_q), \cdot) \) is transversal to \( W \) on \( U_i \) is open and dense in \( \mathbb{R}^q \). Careful reading of the proof of the lemma shows that \( \mathbb{R}^q - A_g \) is a set of measure zero.

In particular, \( \mathbb{R}^q - A_0 = (0)^{-1}(CT_i \cap S) \) and \( \mathbb{R}^q - A_{-g} = \hat{0}^{-1}((CT_i + g) \cap S) \). That is \( CT_i \) and all its translates have measure zero. Thus \( CT_i \) must be a set of Haar measure zero.

**Remarks.** 1. The same proof shows that if \( W \) is a submanifold of \( J^k(B) \), then the elements of \( G^k+1 \) that are transversal to \( W \) are the complement of a
set of Haar measure zero. Recall that $G^{k+1}$ is the set of $C^{k+1}$ sections of $B$ in the Whitney $C^{k+1}$ topology.

2. If we are considering the set of $C^\infty$ maps from compact connected $M$ to $N$ in the $C^\infty$ topology, these form a Fréchet manifold (see [2, III.1.11]). Given $f \in C^\infty(M, N)$, we can find a neighborhood $U$ of $f$ in $C^\infty(M, N)$ that can be identified with an open subset of a Fréchet space and show that those elements of $U$ transversal to $W$ on $M$ are the complement of a set of Haar measure zero.

3. Even if $M$ is noncompact, at first it seems that the definition of Haar measure zero still applies to sections of $C^\infty(B)$ where $B$ is a vector bundle over $M$. In this case, Christensen's proof that countable union of measure zero sets has measure zero does not hold. Also, if $M$ is noncompact, $C^\infty(B)$ is not separable and the generalization of Rademacher's theorem may not hold.

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References


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