ON GENERICITY AND COMPLEMENTS OF MEASURE ZERO SETS IN FUNCTION SPACES

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Abstract. Generic properties of function spaces have been of particular interest in dynamical systems and singularity theory. The underlying assumption has been that the complement of a dense $G_δ$ set is sparse enough to be considered unlikely. Nevertheless, in infinite dimensional spaces, even dense $G_δ$'s may have measure zero. Since there is no one canonical measure on an infinite dimensional Fréchet space, notions of measure zero have not often been considered. Here we use a notion of Haar measure zero on abelian Polish groups due to Christensen [1]. We show that those sections of a finite dimensional vector bundle over a compact manifold whose jets are transverse to a submanifold of the jet bundle are complements of sets of Haar measure zero.

An abelian Polish group $G$ is an abelian topological group with group operation $+$ such that $G$ has a separable Hausdorff topology and such that there exists at least one complete metric on $d$ inducing the given topology.

A universally measurable subset of $G$ is a subset of $G$ that is measurable for every probability measure defined on the Borel sets of $G$.

We will say that a universally measurable subset $A$ of $G$ has Haar measure zero if there is a (nonunique) probability measure $du$ on $G$, called a testing measure, such that for any $g \in G$, $\int_G \chi_{A + g} \, du = 0$. Here $A + g$ is the $g$-translate of $A$, and for any $S \subset G$, $\chi_S$ is the characteristic function of $S$. That is, the set $A$ has Haar measure zero if $A$ and all its translates have measure zero with respect to the testing measure.

We observe without proof that if $G$ were a locally compact group with Haar measure $dh$, then the notion we have defined is equivalent to saying that $\int_G \chi_A \, dh = 0$. We also observe that the countable union of Haar measure zero sets is a set of Haar measure zero. The justification for interest in this notion of measure zero is that it seems to be a suitable one for doing calculus on abelian Polish groups. In particular, there is a generalization of a theorem of Rademacher saying that Lipschitz mappings between certain abelian Polish groups have directional derivatives linear in the direction a.e. in this sense.

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Let $M$ be a compact, connected $C^\infty$ manifold. Let $B$ be a vector bundle over $M$ with finite dimensional fiber. We will use the symbol $C^\infty(B)$ to denote the set of $C^\infty$ sections of $B$ with addition of sections as the group operation. For any nonnegative integer $k$, we will let $J^k(B)$ denote the $k$-jet bundle of sections from $M$ to $B$. There is a well-known transversality theorem that says that if $W$ is a submanifold of $J^k(B)$, then there is a residual subset $R$ of $G$ such that if $g \in R$ then $j^k g: M \to J^k(B)$ is transversal to $W$ (see [2, II. 4.9]). Here $j^k g$ is the $k$-jet mapping of $g$. Our main result will be

I. **Theorem.** Let $G = C^\infty(B)$. Let $W$ be a submanifold of $J^k(B)$ and let $R$ be the subset of those sections $g \in G$ such that $j^k(g): M \to J^k(B)$ is transverse to $W$. Then $G$ is an abelian Polish group and $R$ is the complement of a set of Haar measure zero in $G$.

To prove the theorem, we will need some lemmas.

II. **Lemma.** $G$ is separable.

**Proof of Lemma II.** Since $M$ is compact, we can find a positive integer $p$ and for each $i = 1, 2, \ldots, p$ we can find a $C^\infty$ function $\phi_i: M \to [0, 1]$ such that

1. (i) $\phi_i$ has compact support $C_i$,
   
   (ii) $C_i \subset U_i \subset M$, where $U_i$ is open and diffeomorphic to an open subset of $\mathbb{R}^m$, and
   
   (iii) $\sum_{i=1}^p \phi_i(x) = 1$ for any $x \in M$.

As a result, any section $g \in G$ can be written as a finite sum $g = \sum \phi_i g$. Here $\phi_i g$ is a $C^\infty$ section of $B$ with compact support in $C_i$.

We will show the existence of a countable set of sections that are dense in the set $G_i = \{ \phi_i g: g \in G \}$. This will complete the proof of Lemma II.

We may, without loss of generality, assume that the restriction of $B$ to $U_i$ is trivial. That is, $B|_{U_i} \simeq U_i \times \mathbb{R}^n$ for some integer $n$. In particular, we may identify $U_i$ with an open subset of $\mathbb{R}^m$ and identify sections of $B|_{U_i}$ with $C^\infty$ maps from $U_i$ to $\mathbb{R}^n$. By the Weierstrass approximation theorem, see [3], those maps whose component functions are polynomials with rational coefficients can be used to uniformly approximate any other map up to order $k$ in the $C^k$ sup norm on $C_i$. In particular, if we denote those countably many polynomial maps by $\{ p_j: j \in N \}$, then $\{ \phi_i p_j: j \in N \}$ is dense in $G_i$. This proves the lemma.

II. **Lemma.** There is a complete metric $d$ on $G$ that induces the Whitney $C^\infty$ topology.

**Proof.** The Whitney $C^\infty$ topology is the coarsest topology which is finer than the Whitney $C^k$ topology for any finite $k$. If $G^k = C^k(B)$ is the set of $C^k$ sections of $B$ in the Whitney $C^k$ topology, then there is a complete metric $d^k$ on $G^k$ (see [2, p. 43]). Since $G \subset G^k$, there is an induced metric on $G$ which we will also call $d^k$. If $g, g' \in G$, we will define a metric on $G$ by
It is not hard to see that $d$ is complete and induces the proper topology on $G$.

Now let us prove Theorem I. As in [2, II. 4.9], we choose a countable open cover of $W$ by open sets $W_1, W_2, \ldots$ such that:

2. (i) the closure of $W_i$ in $J^k(B)$ is contained in $W$;

(ii) $\overline{W_i}$ is compact;

(iii) there is a chart $U_i$ on $M$ such that $B|_{U_i} \approx U_i \times \mathbb{R}^n$ for some integer $n$ and such that $\pi(\overline{W_i}) \subset U_i \times \mathbb{R}^n$, where $\pi: J^k(B) \rightarrow B$ is the natural projection mapping; and

(iv) $U_i$ is compact.

If we let $T_i = \{g \in G: j^kg \uparrow W\}$, then just as in [2], each $T_i$ is open and dense and $R = \bigcap T_i$. Let us show that for each $i$, the set $CT_i$, the complement of $T_i$, is a set of Haar measure zero.

Let $\pi_i$ be the projection of $J^k(B)$ onto $M$. Note that $W_i \subset \pi^{-1}_i(U_i)$. We can always assume that we have chosen the $W_i$ and $U_i$ small enough so that there exist a finite integer $q$ and $q$ sections $g_1, g_2, \ldots, g_q \in G$ such that:

3. (i) for each $g \in G$ we can define a map $\bar{g}: \mathbb{R}^q \rightarrow G$ by $\bar{g}(s_1, \ldots, s_q)(x) = (s_1g_1 + \cdots + s_qg_q + g)(x)$;

(ii) the map $\bar{g}$ from $\mathbb{R}^q \times M \rightarrow J^k(B)$, defined by $\bar{g}((s_1, \ldots, s_q), x) = j^k(\bar{g}(s_1, \ldots, s_q))(x)$, is onto $\pi_1^{-1}(U_i)$;

(iii) for each fixed $x \in U_i$, the map $\bar{g}(\cdot, x)$ from $\mathbb{R}^q$ to $J^k(B)$ is surjective at each point in its domain.

To see that such $g_1, \ldots, g_q$ can exist, we identify $U_i$ with a neighborhood of zero in $\mathbb{R}^m$ and let $g_1, \ldots, g_q$ be the set of all distinct maps to $\mathbb{R}^n$ with only one nonzero entry that is a monomial of order $< k$ in $m$ variables.

Let $S = \bar{0}(\mathbb{R}^q)$ be the image of $\mathbb{R}^q$ in $G$. Here 0 is the zero section of $B$. We will induce a probability measure on $S$ by using a probability measure on $\mathbb{R}^q$ which assigns probability zero to sets of Lebesgue measure zero. If $R \subset S$, we define the measure of $R$ to be the measure of $\bar{0}^{-1}(R)$. If $R \subset G$, but $R$ is not a subset of $S$, we define the measure of $R$ to be the measure of $R \cap S$.

We wish to show that $CT_i$ has measure zero with respect to the testing measure we have just defined.

By hypotheses 3(i)–(iii), for any $g \in G$, the map $g: \mathbb{R}^q \times M \rightarrow J^k(B)$ is transversal to $W$ on $U_i$. By Lemma II.4.6 of [2], the set $A_g$ of points $(s_1, \ldots, s_q) \in \mathbb{R}^q$ such that $g((s_1, \ldots, s_q), \cdot)$ is transversal to $W$ on $U_i$ is open and dense in $\mathbb{R}^q$. Careful reading of the proof of the lemma shows that $\mathbb{R}^q - A_g$ is a set of measure zero.

In particular, $\mathbb{R}^q - A_0 = (\bar{0})^{-1}(CT_i \cap S)$ and $\mathbb{R}^q - A_{-g} = \bar{0}^{-1}[((CT_i + g) \cap S)] = (-\bar{g})^{-1}(CT_i \cap S)$. That is $CT_i$, and all its translates have measure zero. Thus $CT_i$ must be a set of Haar measure zero.

Remarks. 1. The same proof shows that if $W$ is a submanifold of $J^k(B)$, then the elements of $G^{k+1}$ that are transversal to $W$ are the complement of a

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set of Haar measure zero. Recall that $G^{k+1}$ is the set of $C^{k+1}$ sections of $B$ in the Whitney $C^{k+1}$ topology.

2. If we are considering the set of $C^\infty$ maps from compact connected $M$ to $N$ in the $C^\infty$ topology, these form a Fréchet manifold (see [2, III.1.11]). Given $f \in C^\infty(M, N)$, we can find a neighborhood $U$ of $f$ in $C^\infty(M, N)$ that can be identified with an open subset of a Fréchet space and show that those elements of $U$ transversal to $W$ on $M$ are the complement of a set of Haar measure zero.

3. Even if $M$ is noncompact, at first it seems that the definition of Haar measure zero still applies to sections of $C^\infty(B)$ where $B$ is a vector bundle over $M$. In this case, Christensen's proof that countable union of measure zero sets has measure zero does not hold. Also, if $M$ is noncompact, $C^\infty(B)$ is not separable and the generalization of Rademacher's theorem may not hold.

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