ALGEBRAIC QUOTIENTS OF BERGMAN DOMAINS

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Abstract. A discrete group of automorphisms of a domain in \( \mathbb{C}^n \) is constructed so that the quotient is algebraic.

A Bergman domain in \( \mathbb{C}^n \) is a set \( D = \{(z_1, \ldots, z_n)/z_1 \in U, z_{i+1} \in B_{i+1}(z_1, \ldots, z_i), i = 1, \ldots, n - 1\} \) where \( B_{i+1} \) is a Jordan domain in \( \mathbb{C} \) whose boundary admits a parametric representation

\[
\zeta_{i+1} = W(z_1, \ldots, z_i; x), \quad x \in \mathbb{R}.
\]

For fixed \( x \) \( W \) is a holomorphic function of \( (z_1, \ldots, z_i) \in D_i \), the projection of \( D \) onto the first \( i \) coordinates.

Given an arbitrary quasi-projective algebraic variety \( A_0 \) there is a Zariski open subset \( A_1 \subset A_0 \) such that the universal covering of \( A_1 \) is a Bergman domain (cf. Griffiths \[2\], Bers \[1\]). In this note we prove a partial converse to the theorem above, namely that a discrete group of analytic automorphisms of \( D \), obtained as an extension of certain Fuchsian groups of finite type, gives as quotient an algebraic variety.

Let \( \mu \) be a measurable function in the upper half plane \( U \) with \( \| \mu \|_{\infty} < 1 \). We denote by \( W^\mu \) the unique homeomorphic solution of the Beltrami equation

\[
\frac{\partial W^\mu}{\partial \bar{z}} = \mu \frac{\partial W^\mu}{\partial z}
\]

fixing 0, 1 and \( \infty \), where \( \mu \) is extended by 0 to the lower half plane. The domains considered here are to satisfy the following additional property:

For any point \( \zeta \in D_i \) there is a neighborhood \( \zeta \in V \subset D_i \) and a holomorphic function \( \alpha_i: V \to L^\infty(U) \) such that \( B_{i+1} = W^{\alpha_i(z)}(U) \) for all \( z \in V \).

Consider an ascending chain of groups

\[
\{1\} = G_0 \supset G_1 \supset \cdots \supset G_n = G
\]

where \( G_{i+1} \) is a split extension \( G_i \) by a Fuchsian group \( H_i \) (0 \( \leq i \leq n - 1 \)). Assume further that there is an analytic function \( f_i: D_i \to T(H_i) \) into the Teichmüller space of \( H_i \), "coinciding" with \( \alpha_i \) in each neighborhood \( V \), such that \( f_i \) maps \( G_i \) into a subgroup of the Teichmüller modular group \( \text{Mod}(H_i) \).

It is then possible (cf. Riera \[4\]) to define by recurrence an action of \( G_{i+1} \) on \( D_{i+1} \) by means of the formula

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(g, h)(z, \xi) = \left( g(z), W^{f(z)}(z) \circ h \circ \delta_i(g) \circ \left( W^{f(z)}(z) \right)^{-1}(\xi) \right)

for all z \in D_i, \xi \in B_{i+1}, g \in G_i, h \in H_i, where \delta_i is an appropriate homomorphism from G_i into a group mod(H_i).

**Theorem.** Let G be a group of analytic automorphisms of a Bergman domain D, obtained by a chain of split extensions of Fuchsian groups H_i of finite type (P_i, K_i), 0 < i < n - 1. If K_i = 0 for 0 < i < n - 2, then D/G is a quasi-projective algebraic variety.

**Proof.** If \( \phi_i, \psi_i \) denote fundamental regions for G_i and H_i respectively, then a fundamental region for G_{i+1} on D_{i+1} is the set

\[ \phi_{i+1} = \left\{ (z, \xi) : z \in \phi_i, \xi \in W^{f(z)}( \psi_i ) \right\}. \]

It follows from the assumptions that the fundamental region of G_{n-1} is compact; i.e., \( D_{n-1}/G_{n-1} \) is a compact manifold. The Bergman metric on \( D_{n-1} \) projects to the quotient and gives a Hodge metric; therefore \( D_{n-1}/G_{n-1} \) is an algebraic variety (cf. Kodaira [3]).

The quotient \( U/H_{n-1} = S \) is a compact Riemann surface \( \tilde{S} \) from which \( K_{n-1} \) points \( \alpha_1, \ldots, \alpha_{K_{n-1}} \) have been removed. Let \( \tilde{H}_{n-1} \) be a Fuchsian group representing \( \tilde{S} \) and let \( \alpha_1, \ldots, \alpha_{K_{n-1}} \in U \) project to \( \alpha_1, \ldots, \alpha_{K_{n-1}} \) in this representation. Now let \( \pi : U \to U \) be a universal covering map of \( U \setminus \{ g(\alpha_p), 1 < p < K_{n-1}/g \in \tilde{H}_{n-1} \} \), that conjugates \( H_{n-1} \) onto \( \tilde{H}_{n-1} \).

For fixed \( z \in D_{n-1} \) let \mu denote a Beltrami differential in the class \( f(z) \in T(H_{n-1}) \). We project down to \( \tilde{H}_{n-1} \) by the formula

\[ \mu' = \mu \cdot \pi^{-1}/\pi' \]

to obtain a new function \( \tilde{f} : D_{n-1} \to T(\tilde{H}_{n-1}) \) associating \( z \) to \( \tilde{\mu} \). We observe that the points \( W^{f(z)}(\alpha_p), 1 < p < K_{n-1} \), are well defined since they are determined entirely by the class \( f(z) \). Thus we have constructed an extension \( \tilde{G} \) of \( G_{n-1} \) by \( \tilde{H}_{n-1} \) that acts on a domain \( \tilde{D} \). Write \( \tilde{M} = D/G, \tilde{M} = \tilde{D}/\tilde{G} \). Since \( \tilde{M} \) is compact there exist functions \( \phi_1, \ldots, \phi_n \) that embed \( \tilde{M} \) into \( \mathbb{P}^N \) for some \( N \); the image in \( \mathbb{P}^N \) is an algebraic variety \( A_0 \).

Finally, the functions

\[ z \mapsto \left( \phi_1(z, W^{f(z)}(\alpha_p)), \ldots, \phi_N(z, W^{f(z)}(\alpha_p)) \right), \]

\( 1 < p < K_{n-1} \), induce regular mappings from the algebraic variety \( D_{n-1}/G_{n-1} \) into \( A_0 \). The image \( B \) is then a Zariski closed set in \( A_0 \) and \( M \) is isomorphic to \( A_1 = A_0 \setminus B \).

**References**


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