PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 68, Number 3, March 1978

ALGEBRAIC QUOTIENTS OF BERGMAN DOMAINS

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ABSTRACT. A discrete group of automorphisms of a domain in $\mathbf{C}^n$ is constructed so that the quotient is algebraic.

A Bergman domain in $\mathbf{C}^n$ is a set $D = \{(z_1, \ldots, z_n)/z_1 \in U,\ z_{i+1} \in B_{i+1}(z_1, \ldots, z_i),\ i = 1, \ldots, n-1\}$ where B_{i+1} is a Jordan domain in $\mathbf{C}$ whose boundary admits a parametric representation

$$\zeta_{i+1} = W(z_1, \ldots, z_i; x), \qquad x \in \mathbf{R}.$$

For fixed x W is a holomorphic function of $(z_1, \ldots, z_i) \in D_i$, the projection of D onto the first i coordinates.

Given an arbitrary quasi-projective algebraic variety A_0 there is a Zariski open subset $A_1 \subset A_0$ such that the universal covering of A_1 is a Bergman domain (cf. Griffiths **[2]**, Bers **[1]**). In this note we prove a partial converse to the theorem above, namely that a discrete group of analytic automorphisms of D, obtained as an extension of certain Fuchsian groups of finite type, gives as quotient an algebraic variety.

Let μ be a measurable function in the upper half plane U with $\|\mu\|_\infty < 1$. We denote by W^μ the unique homeomorphic solution of the Beltrami equation

$$\partial W^\mu / \partial \bar{z} = \mu \partial W^\mu / \partial z$$

fixing 0, 1 and ∞, where μ is extended by 0 to the lower half plane. The domains considered here are to satisfy the following additional property:

For any point $\mathring{z} \in D_i$ there is a neighborhood $\mathring{z} \in V \subset D_i$ and a holomorphic function α_i: $V \to L_1^\infty(U)$ such that $B_{i+1} = W^{\alpha_i(z)}(U)$ for all $z \in V$.

Consider an ascending chain of groups

$$\{1\} = G_0 \rhd G_1 \rhd \cdots \rhd G_n = G$$

where G_{i+1} is a split extension G_i by a Fuchsian group H_i ($0 \leqslant i \leqslant n-1$). Assume further that there is an analytic function f_i: $D_i \to T(H_i)$ into the Teichmüller space of H_i, "coinciding" with α_i in each neighborhood V, such that f_i maps G_i into a subgroup of the Teichmüller modular group $\operatorname{Mod}(H_i)$.

It is then possible (cf. Riera **[4]**) to define by recurrence an action of G_{i+1} on D_{i+1} by means of the formula

Received by the editors February 17, 1977.

AMS (MOS) subject classifications (1970). Primary 14H30; Secondary 32J20, 32G15.

$$(g, h)(z, \zeta) = \left(g(z), W^{f_i(g(z))} \circ h \circ \delta_i(g) \circ \left(W^{f_i(z)}\right)^{-1}(\zeta)\right)$$

for all $z \in D_i$, $\zeta \in B_{i+1}$, $g \in G_i$, $h \in H_i$, where δ_i is an appropriate homomorphism from G_i into a group $\operatorname{mod}(H_i)$.

THEOREM. *Let G be a group of analytic automorphisms of a Bergman domain D, obtained by a chain of split extensions of Fuchsian groups H_i of finite type (P_i, K_i), $0 \leqslant i \leqslant n-1$. If $K_i = 0$ for $0 \leqslant i \leqslant n-2$, then D/G is a quasi-projective algebraic variety.*

PROOF. If ϕ_i, ψ_i denote fundamental regions for G_i and H_i respectively, then a fundamental region for G_{i+1} on D_{i+1} is the set

$$\phi_{i+1} = \left\{(z, \zeta)/z \in \phi_i, \zeta \in W^{f_i(z)}(\psi_i)\right\}.$$

It follows from the assumptions that the fundamental region of G_{n-1} is compact; i.e., D_{n-1}/G_{n-1} is a compact manifold. The Bergman metric on D_{n-1} projects to the quotient and gives a Hodge metric; therefore D_{n-1}/G_{n-1} is an algebraic variety (cf. Kodaira [3]).

The quotient $U/H_{n-1} = S$ is a compact Riemann surface $\hat{S}$ from which K_{n-1} points $\alpha_1, \ldots, \alpha_{K_{n-1}}$ have been removed. Let $\hat{H}_{n-1}$ be a Fuchsian group representing $\hat{S}$ and let $a_1, \ldots, a_{K_{n-1}} \in U$ project to $\alpha_1, \ldots, \alpha_{K_{n-1}}$ in this representation. Now let π: $U \to U$ be a universal covering map of $U \setminus \{g(a_p), 1 \leqslant p \leqslant K_{n-1}/g \in \hat{H}_{n-1}\}$, that conjugates H_{n-1} onto $\hat{H}_{n-1}$.

For fixed $z \in D_{n-1}$ let μ denote a Beltrami differential in the class $f_{n-1}(z) \in T(H_{n-1})$. We project down to $\hat{H}_{n-1}$ by the formula

$$\mu \overline{\pi'} / \pi' = \mu$$

to obtain a new function $\hat{f}$: $D_{n-1} \to T(\hat{H}_{n-1})$ associating z to $\hat{\mu}$. We observe that the points $W^{\hat{f}(z)}(\alpha_p)$, $1 \leqslant p \leqslant k_{n-1}$, are well defined since they are determined entirely by the class $\hat{f}(z)$. Thus we have constructed an extension $\hat{G}$ of G_{n-1} by $\hat{H}_{n-1}$ that acts on a domain $\hat{D}$. Write $M = D/G$, $\hat{M} = \hat{D}/\hat{G}$. Since $\hat{M}$ is compact there exist functions $\phi_1, \ldots, \phi_n$ that embed $\hat{M}$ into $\mathbf{P}^N$ for some N: the image in $\mathbf{P}^N$ is an algebraic variety A_0.

Finally, the functions

$$z \mapsto \left(\phi_1\left(z, W^{\hat{f}(z)}(a_p)\right), \ldots, \phi_N\left(z, W^{\hat{f}(z)}(a_p)\right)\right),$$

$1 \leqslant p \leqslant K_{n-1}$, induce regular mappings from the algebraic variety D_{n-1}/G_{n-1} into A_0. The image B is then a Zariski closed set in A_0 and M is isomorphic to $A_1 = A_0 \setminus B$.

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