TWO THEOREMS ON THE MAPPING CLASS GROUP OF A SURFACE

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Abstract. The mapping class group of a closed surface of genus \( g > 3 \) is perfect. An infinite set of generators is given for the subgroup of maps that induce the identity on homology.

Let \( T_g \) be the boundary of a 3-ball with \( g \) handles, \( H(T_g) \) be the group of all orientation preserving homeomorphisms of \( T_g \), \( D(T_g) \) be the subgroup of all homeomorphisms which are isotopic to the identity. The mapping class group \( M(T_g) \) is the quotient group \( H(T_g)/D(T_g) \). Generators for \( M(T_g) \) are known, but a complete set of relations for \( M(T_g) \) is only known for \( g = 2 \) [2]. The commutator subgroup \( M(T_2)/[M(T_2), M(T_2)] \) is \( \mathbb{Z}_{10} \).

We shall prove that \( M(T_g) \) is perfect for \( g > 3 \). We shall also give an infinite set of generators for the subgroup \( K_g \) of \( M(T_g) \) which induces the identity on homology, i.e. the kernel of the map from \( M(T_g) \) onto \( \Gamma_g \), Siegel’s modular group [1].

Theorem 1. The mapping class group \( M(T_g) \), for \( g > 3 \), is perfect.

Proof. For \( g > 3 \), it is known [1] that the commutator subgroup of \( M(T_g) \) is either \( \mathbb{Z}_2 \) or the trivial subgroup, and also that, if it were \( \mathbb{Z}_2 \), then every Dehn twist about a nonseparating curve on \( T_g \) would be mapped onto the generator of \( \mathbb{Z}_2 \). To prove that the commutator subgroup of \( M(T_g) \) is trivial, for \( g > 3 \), it is sufficient to exhibit a product of an odd number of Dehn twists about nonseparating curves that is isotopic to the identity. Such a product is given by the 15 following twists:

\[
(Y_3 Y_2 Y_1 Z_1^{-1} U_2 Z_2 U_2^{-1} Y_2 Z_1 U_2 Z_2 Y_2^{-1} U_2^{-1}) A^{-1} = \text{id}
\]

where \( U_i, Y_i, Z_i, A \) are Dehn twists about the curves \( u_i, y_i, z_i, a \). (See Figures 1 and 2.) At the end of the paper, we will explain how to check the assertion.

Let \( K_g \) be the kernel of the map from \( M(T_g) \) to \( \Gamma_g \), Siegel’s modular group [3].

Theorem 2. For \( g > 3 \), \( K_g \) is generated by 2 types of twists:

Type 1. Dehn twists about separating curves (which divide the surface in such a way that one of the resulting components has genus 1 or 2).
Type 2. "double twists", i.e. simultaneous Dehn twists $R$ and $S^{-1}$ about disjoint homologous curves $r$ and $s$ ($r$ and $s$ divide the surface in such a way that one of the resulting components has genus 1).

Proof. Birman showed ([3, Theorem 2]) that, for $g \geq 3$, $K_g$ is generated by the conjugates in $M(T_g)$ of

\[
(Y_1U_1Y_1)^4,
(Y_1U_1Z_1U_2Y_2)^6,
(Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1)Y_1(Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1)^{-1}Y_1^{-1},
(Y_3Y_2Y_1Z_1^{-1}U_2Z_2^{-1}U_2^{-1}Y_2^{-1}U_2Z_1U_2Z_2Y_2^{-1}U_2^{-1})P_2Z_1^{-1}P_2^{-1},
\]

where $P_2 = (Y_2U_2Y_2)(Y_2U_2Z_2U_3Y_3)(Y_2U_2Y_2)^{-1}$. 

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The conjugates in $M(T_g)$ of twists about curves are the twists about the images of the curves under the homeomorphism by which you conjugate ([4, Lemma 1]). All twists about separating curves which divide the surface in such a way that one of the components is of genus 1 (respectively genus 2) are conjugate and all "double twists" of type 2 are conjugate. Therefore it suffices to show that the above homeomorphisms are of type 1 or type 2.

Let $A, B, C, D, E, F$ be Dehn twists about $a, b, c, d, e, f$. We assert that the following equations hold:

\[
\begin{align*}
(Y_1U_1Y_1)^4 &= B, \\
(Y_1U_1Z_1U_2Y_2)^6 &= C, \\
(Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1)Y_1 (Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1)^{-1}Y_1^{-1} &= D Y_1^{-1}, \\
(Y_3Y_2Y_1Z_1^{-1}U_2Z_2^{-1}U_2^{-1}Y_2^{-1}U_2Z_1U_2Z_2Y_2^{-1}U_2^{-1})P_2Z_1^{-1}P_2^{-1} &= AE^{-1} = (AF^{-1})(FE^{-1}).
\end{align*}
\]
The twists $B$ and $C$ are of Type 1. The twists $DY_1^{-1}$, $AF^{-1}$, $FE^{-1}$ are of Type 2.

One may verify the assertions in Theorems 1 and 2 by lengthy calculations. It suffices to show that the images of generators of $\Pi_1(T_g)$ under the sequences of twists are isotopic for each side of the equalities. This type of calculation is illustrated by Lickorish ([4, p. 773]). Note that it is necessary to keep the base point fixed because there are nontrivial homeomorphisms of $M(T_g)$ such that the image of each generator of $\Pi_1(T_g)$ is freely homotopic to the generator.

REFERENCES