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TWO THEOREMS ON THE MAPPING CLASS GROUP OF A SURFACE

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ABSTRACT. The mapping class group of a closed surface of genus > 3 is perfect. An infinite set of generators is given for the subgroup of maps that induce the identity on homology.

Let T_g be the boundary of a 3-ball with g handles, $H(T_g)$ be the group of all orientation preserving homeomorphisms of T_g, $D(T_g)$ be the subgroup of all homeomorphisms which are isotopic to the identity. The mapping class group $M(T_g)$ is the quotient group $H(T_g)/D(T_g)$. Generators for $M(T_g)$ are known, but a complete set of relations for $M(T_g)$ is only known for $g = 2$ [**2**]. The commutator subgroup $M(T_2)/[M(T_2), M(T_2)]$ is $\mathbf{Z}_{10}$.

We shall prove that $M(T_g)$ is perfect for $g \geqslant 3$. We shall also give an infinite set of generators for the subgroup K_g of $M(T_g)$ which induces the identity on homology, i.e. the kernel of the map from $M(T_g)$ onto Γ_g, Siegel's modular group [**1**].

THEOREM 1. *The mapping class group $M(T_g)$, for $g \geqslant 3$, is perfect.*

PROOF. For $g \geqslant 3$, it is known [**1**] that the commutator subgroup of $M(T_g)$ is either $\mathbf{Z}_2$ or the trivial subgroup, and also that, if it were $\mathbf{Z}_2$, then every Dehn twist about a nonseparating curve on T_g would be mapped onto the generator of $\mathbf{Z}_2$. To prove that the commutator subgroup of $M(T_g)$ is trivial, for $g \geqslant 3$, it is sufficient to exhibit a product of an odd number of Dehn twists about nonseparating curves that is isotopic to the identity. Such a product is given by the 15 following twists:

$$(Y_3Y_2Y_1Z_1^{-1}U_2Z_2^{-1}U_2^{-1}Y_2^{-1}U_2Z_1U_2Z_2Y_2^{-1}U_2^{-1})A^{-1} = \mathrm{id}$$

where U_i, Y_i, Z_i, A are Dehn twists about the curves u_i, y_i, z_i, a. (See Figures 1 and 2.) At the end of the paper, we will explain how to check the assertion.

Let K_g be the kernel of the map from $M(T_g)$ to Γ_g, Siegel's modular group [**3**].

THEOREM 2. *For $g \geqslant 3$, K_g is generated by* 2 *types of twists*:

TYPE 1. *Dehn twists about separating curves* (*which divide the surface in such a way that one of the resulting components has genus* 1 *or* 2).

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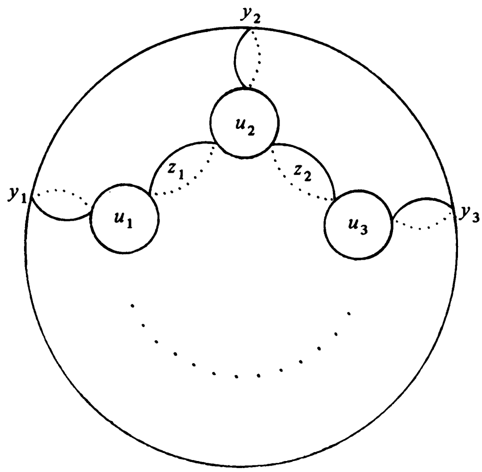

FIGURE 1

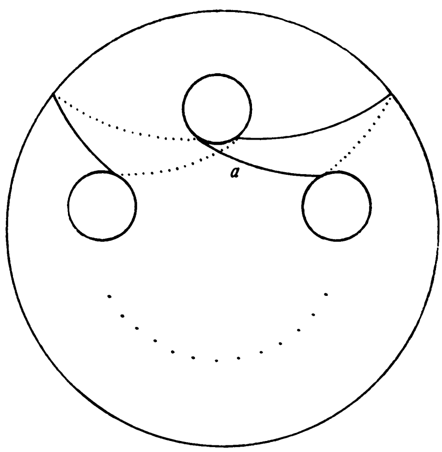

FIGURE 2

TYPE 2. "*double twists*", *i.e. simultaneous Dehn twists R and S^{-1} about disjoint homologous curves r and s (r and s divide the surface in such a way that one of the resulting components has genus* 1).

PROOF. Birman showed ([3, Theorem 2]) that, for $g \geqslant 3$, K_g is generated by the conjugates in $M(T_g)$ of

$$(Y_1U_1Y_1)^4,$$
$$(Y_1U_1Z_1U_2Y_2)^6,$$
$$(Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1)Y_1(Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1)^{-1}Y_1^{-1},$$
$$(Y_3Y_2Y_1Z_1^{-1}U_2Z_2^{-1}U_2^{-1}Y_2^{-1}U_2Z_1U_2Z_2Y_2^{-1}U_2^{-1})P_2Z_1^{-1}P_2^{-1},$$
$$\text{where } P_2 = (Y_2U_2Y_2)(Y_2U_2Z_2U_3Y_3)^3(Y_2U_2Y_2)^{-1}.$$

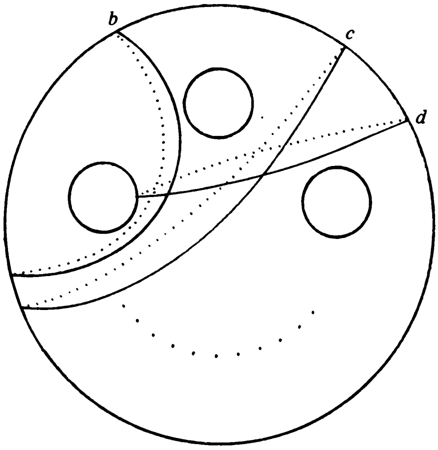

FIGURE 3

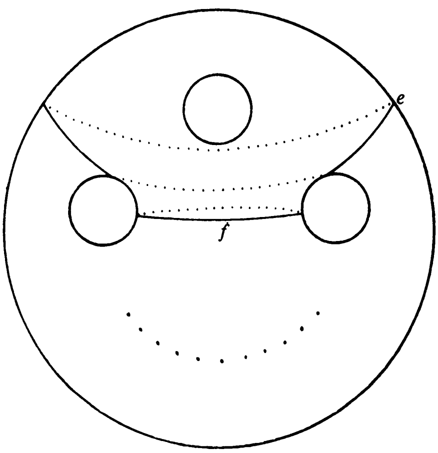

FIGURE 4

The conjugates in $M(T_g)$ of twists about curves are the twists about the images of the curves under the homeomorphism by which you conjugate ([**4**, Lemma 1]). All twists about separating curves which divide the surface in such a way that one of the components is of genus 1 (respectively genus 2) are conjugate and all "double twists" of type 2 are conjugate. Therefore it suffices to show that the above homeomorphisms are of type 1 or type 2.

Let A, B, C, D, E, F be Dehn twists about a, b, c, d, e, f. We assert that the following equations hold:

$$(Y_1U_1Y_1)^4 = B,$$
$$(Y_1U_1Z_1U_2Y_2)^6 = C,$$
$$(Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1)Y_1(Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1)^{-1}Y_1^{-1} = DY_1^{-1},$$
$$(Y_3Y_2Y_1Z_1^{-1}U_2Z_2^{-1}U_2^{-1}Y_2^{-1}U_2Z_1U_2Z_2Y_2^{-1}U_2^{-1})P_2Z_1^{-1}P_2^{-1}$$
$$= AE^{-1} = (AF^{-1})(FE^{-1}).$$

The twists B and C are of Type 1. The twists DY_1^{-1}, AF^{-1}, FE^{-1} are of Type 2.

One may verify the assertions in Theorems 1 and 2 by lengthy calculations. It suffices to show that the images of generators of $\Pi_1(T_g)$ under the sequences of twists are isotopic for each side of the equalities. This type of calculation is illustrated by Lickorish ([**4**, p. 773]). Note that it is necessary to keep the base point fixed because there are nontrivial homeomorphisms of $M(T_g)$ such that the image of each generator of $\Pi_1(T_g)$ is freely homotopic to the generator.

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