A COUNTEREXAMPLE IN THE FACTORIZATION OF BANACH SPACE OPERATORS

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Abstract. A counterexample is given which completes the Banach space generalization of a theorem of R. G. Douglas concerning the factorization of Hilbert space operators.

In [1] R. G. Douglas proves the equivalence of three conditions related to factoring a Hilbert space operator. In [2] Mary Embry determines all but one of the possible implications among those three conditions interpreted for operators on a Banach space. This short note gives a counterexample to show that the one remaining possible implication does not hold.

In the notation of [4] define $A$ and $B$ on $(c_0)$ by $Ae_k = 0$ for $k \neq 1$ and $Ae_1 = y = (2^{-1}, 2^{-2}, \ldots)$ and $B(x_1, x_2, \ldots) = (2^{-1}x_1, 2^{-2}x_2, \ldots)$. Recall $(c_0)' = l^1$ and $(l^1)' = c_0$. The straightforward proof of the next lemma is omitted.

Lemma. The dual operator $B''$ on $l^\infty$ maps $(f_1, f_2, \ldots)$ to $(2^{-1}f_1, 2^{-2}f_2, \ldots)$ and the images of $B$ and $B''$ are $\{ (x_1, x_2, \ldots) \in (c_0) : \lim 2^n x_n = 0 \}$ and $\{ (h_1, h_2, \ldots) \in l^\infty : \sup |2^nh_n| < \infty \}$, respectively. The operator $A''$ on $l^\infty$ is defined by $A''(g_1, g_2, \ldots) = g_1h$, where $h = (2^{-1}, 2^{-2}, \ldots)$ and the images of $A$ and $A''$ are span{y} and span{h}, respectively.

Theorem. Condition (i) below holds but (ii) does not hold:

(i) for some $c > 0$, $\| A'f \| < c \| B'f \|$ for all $f \in l^1$;

(ii) the image of $B$ contains the image of $A$—i.e. $B(c_0) \supseteq A(c_0)$.

Proof. From the Lemma it follows that $A''l^\infty \subseteq B''l^\infty$ and $A(c_0) \nsubseteq B(c_0)$. Theorem 1 of [2] implies that (i) above holds.

This counterexample fills a gap in [3] showing that the converse to Theorem 1 of [3] does not hold.

References


Received by the editors June 13, 1977.