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APPROXIMABILITY OF THE INVERSE OF AN OPERATOR

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ABSTRACT. Let A be an invertible operator on a complex Hilbert space $\mathcal{H}$. A necessary and sufficient condition is given for A^{-1} to be a weak limit of polynomials in A.

Let A be a bounded linear invertible operator on a complex Hilbert space $\mathcal{H}$. If $\mathcal{H}$ is finite dimensional then there exists a polynomial P such that $A^{-1} = p(A)$. The expected infinite dimensional analogue is false. If $\mathcal{H}$ is infinite dimensional then A^{-1} is not in general a limit of polynomials in A even in the weak operator topology; a simple example is the bilateral shift. Sufficient conditions for A^{-1} to be a weak (equivalently, strong) limit of polynomials in A were given in [2] and [3]. Here a necessary and sufficient condition is presented in the spirit of Theorem 1 of [2]. The proof was motivated by some results given in [1].

Let $\mathfrak{R}$ denote the weak closure of the algebra of polynomials in A, $\mathcal{H}^{(n)}$ the usual direct sum of n copies of $\mathcal{H}$, $A^{(n)}$ the operator on $\mathcal{H}^{(n)}$ defined by

$$A^{(n)}\langle x_1, \ldots, x_n\rangle = \langle Ax_1, \ldots, Ax_n\rangle$$

and $\mathfrak{R}^{(n)} = \{T^{(n)}: T \in \mathfrak{R}\}$. Lat A will denote the lattice of closed invariant subspaces of A. The characterization of $\mathfrak{R}$ given in Lemma 1 is standard [5, p. 118].

LEMMA 1. *$T \in \mathfrak{R}$ if and only if* Lat $A^{(n)} \subseteq$ Lat $T^{(n)}$ *for all $n \geq 1$.*

DEFINITION. The operator T is strictly positive if there exists a real number $\delta > 0$ such that for all $f \in \mathcal{H}$, $\mathrm{Re}(Tf, f) \geq \delta \|f\|^2$.

LEMMA 2. *If T is strictly positive, so is $T^{(n)}$ for $n \geq 1$.*

PROOF. Left to the reader.

THEOREM. *Let A be an invertible operator on $\mathcal{H}$, $\mathfrak{R}$ the weak closure of the algebra of polynomials in A. Then $A^{-1} \in \mathfrak{R}$ if and only if there exists $T \in \mathfrak{R}$ such that $T^{-1} \in \mathfrak{R}$ and TA is strictly positive.*

PROOF. If $A^{-1} \in \mathfrak{R}$ just pick $T = A^{-1}$. Suppose $T, T^{-1} \in \mathfrak{R}$ and TA is strictly positive. We show Lat $A \subset$ Lat A^{-1}. Let $\mathfrak{M} \in$ Lat A and P be the

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orthogonal projection on $\mathfrak{M}$. We will denote by PSP the operator $PS|\mathfrak{M}$. Then to show $\mathfrak{M} \in \operatorname{Lat} A^{-1}$, it suffices to show that PAP is invertible for if this is the case, $\mathfrak{M} = PAP\mathfrak{M} = A\mathfrak{M}$ (since $\mathfrak{M} \in \operatorname{Lat} A$) and thus $A^{-1}\mathfrak{M} = \mathfrak{M}$. Since $T, T^{-1} \in \mathfrak{R}$, PTP is invertible with inverse $PT^{-1}P$. Thus the invertibility of PAP is equivalent to the invertibility of $PTAP = PTPAP$. We show $PTAP$ is invertible.

By our assumption that TA is strictly positive, there is some $\delta > 0$ such that $\operatorname{Re}(TAf, f) \geqslant \delta \|f\|^2$ for all f in $\mathfrak{H}$. Applied in particular to f in $\mathfrak{M}$ this shows that $PTAP$ is strictly positive and thus invertible. Therefore $\operatorname{Lat} A \subset \operatorname{Lat} A^{-1}$.

We complete the proof by noting that the same argument applies to show $\operatorname{Lat} A^{(n)} \subseteq \operatorname{Lat} A^{-1(n)}$ and applying Lemma 1.

Let $\omega(T)$ denote the numerical range of $T = \{(Tx, x): \|x\| = 1\}$.

COROLLARY. *$A^{-1} \in \mathfrak{R}$ if and only if there exists $T \in \mathfrak{R}$ such that $T^{-1} \in \mathfrak{R}$ and $0 \notin \omega(TA)$.*

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