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A CROSS SECTION THEOREM AND AN APPLICATION TO C^*-ALGEBRAS

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ABSTRACT. The purpose of this note is to prove a cross section theorem for certain equivalence relations on Borel subsets of a Polish space. This theorem is then applied to show that cross sections always exist on countably separated Borel subsets of the dual of a separable C^*-algebra.

See Auslander-Moore [2], Bourbaki [3], Kuratowski [9], and Mackey [12] for the main results and notation in Polish set theory used in this paper.

The main result of this note is the following theorem.

THEOREM 1. *Let B be a Borel subset of the Polish space X. Let R be an equivalence relation on B such that each R-equivalence class is both a G_δ and an F_σ in X, and such that the R-saturation of each relatively open subset of B is Borel. Then the quotient Borel space B/R is standard, and there is a Borel cross section $f: B/R \to B$ for R.*

Notice that if the R-saturation of each relatively closed subset of B is Borel, then the R-saturation of each relatively open subset of B is Borel, for each relatively open subset of B is the countable union of relatively closed sets.

A number of preliminary lemmas are proved first.

LEMMA 2. *Let (Y, d) be a separable metric space and let R be an equivalence relation on Y such that the R-saturation of each open set is Borel. Then there is a Borel set S whose intersection with each R-equivalence class which is complete with respect to d is nonempty, and whose intersection with each R-equivalence class is at most one point.*

PROOF. By the proofs (but not the statements) of Theorem 4, p. 206, Bourbaki [3] and Lemme 2, p. 279, Dixmier [4], there exists a decreasing sequence of Borel subsets of Y, say S_n, so that $S_n \cap R(y) \neq \emptyset$, diameter$(S_n \cap R(y)) \to 0$, and $\bigcap_{n \geq 1}(S_n \cap R(y)) = \bigcap_{n \geq 1}(\overline{(S_n \cap R(y))} \cap R(y))$ for each y in Y. Let $S = \bigcap_{n \geq 1} S_n$. S is a Borel subset of Y, the intersection of S with each complete R-equivalence class is nonempty, and the intersection of S with each R-equivalence class is at most one point. Q.E.D.

LEMMA 3. *Let Y be a Polish space and D a subset of Y which is both a G_δ and*

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an F_σ. Then there is an open set V in Y so that $D \cap V$ is nonempty and $D \cap V$ is closed in V.

PROOF. We may assume that $\bar{D} = Y$. Since D is a G_δ, $Y - D$ is a countable union of closed sets. None of these closed sets has an interior, for D is dense in Y. But D is a countable union of closed sets. The Baire category theorem implies that one of these closed sets has an interior. Hence, there exists an open set V so that V is contained in D. Q.E.D.

LEMMA 4. *Let X and R be as in Theorem 1. Then X/R is countably separated.*

PROOF. Let V_m $(m \geqslant 1)$ be a basis for the topology of X. Then the $R(V_m)$ $(m \geqslant 1)$ are Borel sets which separate the R-equivalence classes. To see this, let a and b be elements of X so that $R(a)$ and $R(b)$ are disjoint. If $R(b)$ is not contained in $\overline{R(a)}$, there exists a c in $R(b)$ and a positive integer m so that c is in V_m and $V_m \cap \overline{R(a)} = \varnothing$. But then $V_m \cap R(a) = \varnothing$, and so $R(V_m) \cap R(a)$ is empty. Hence, $R(b)$ is contained in $R(V_m)$ and $R(a)$ is contained in $X - R(V_m)$. So we may assume that $R(b)$ is contained in $\overline{R(a)}$. It follows from Lemma 3 that there is an integer m so that $R(a) \cap V_m$ is nonempty and $R(a) \cap V_m = \overline{R(a)} \cap V_m$. But then $R(V_m) \cap R(b)$ is empty. If not, there exists a c in $R(b) \cap V_m \subseteq \overline{R(a)} \cap V_m = R(a) \cap V_m$. Hence, $R(b) = R(c) = R(a)$. Contradiction. Hence, $R(a)$ is contained in $R(V_m)$ and $R(b)$ is contained in $X - R(V_m)$. Q.E.D.

PROOF OF THEOREM 1. If V is an open subset of X and $U = B \cap V$, R defines an equivalence relation R_U on U by $R_U(b) = R(b) \cap U$. The R_U-saturation of any open set is Borel. Now V itself is a Polish space, and each R_U-equivalence class is both a G_δ and an F_σ in V. Hence, by Lemma 2, there is a Borel set S which intersects each R_U-equivalence class in at most one point, and which intersects each R_U-equivalence class which is closed in V in exactly one point. Let V_m $(m \geqslant 1)$ be a basis for the topology of X. For each V_m, let S_m be a corresponding S, and let $S' = \bigcup_{m \geqslant 1} S_m \cdot S'$ is a Borel subset of X. S' intersects each R-equivalence class in at most countably many points. Furthermore, S' intersects each R-equivalence class in at least one point by Lemma 3. X/R is countably separated, and therefore is Borel isomorphic to an analytic subset of [0, 1] by Proposition 2.9, p. 8, of Auslander-Moore [2]. Let $g\colon S' \to X/R$ be the natural surjective Borel mapping. The graph of g, say C, is a Borel subset of $S' \times [0, 1]$. Horizontal sections of C are at most countable. Hence, theorems of S. Braun and N. Luzin (see 42.4.5, p. 378, and 42.5.3, p. 381, of Hahn [8]) show that the horizontal projection of C, namely X/R, is standard and that there exists a Borel subset S'' of S' so that $g|S''$ is a bijection onto X/R. Let $f = (g|S'')^{-1}$. f is a Borel mapping by Souslin's theorem. Q.E.D.

See Dixmier [5] for most of the notation and results on C^*-algebras used in this note. The Borel structure on the dual of a C^*-algebra is that generated by the hull-kernel topology. The following corollary might be a useful tool in proving local versions of known theorems in C^*-algebras and group representations (see, for instance, Moore's appendix to Auslander and Kostant [1]).

Corollary 5. *Let $\mathfrak{A}$ be a separable C^*-algebra and let B be a Borel subset of $\hat{\mathfrak{A}}$ whose relative Borel structure separates points. Then B is standard, and there is a Borel cross section f: $B \to \mathrm{Irr}(\mathfrak{A})$.*

Proof. Let p: $\hat{\pi} \to \mathrm{kernel}(\hat{\pi})$, $\hat{\mathfrak{A}} \to \mathrm{Prim}(\mathfrak{A})$, be the natural open mapping. From the definition of the topology of $\hat{\mathfrak{A}}$, U is open in $\hat{\mathfrak{A}}$ if and only if $p(U)$ is open in $\mathrm{Prim}(\mathfrak{A})$, in which case $U = p^{-1}(p(U))$. Now consider the set $\mathcal{S}$ of all subsets B of $\hat{\mathfrak{A}}$ such that $B = p^{-1}(p(B))$. $\mathcal{S}$ is clearly closed under countable unions, and $\mathcal{S}$ is closed under complements since p is surjective. Since $\mathcal{S}$ contains the open subsets of $\hat{\mathfrak{A}}$, $\mathcal{S}$ therefore contains all Borel subsets of $\hat{\mathfrak{A}}$. Therefore, B is a Borel subset of $\hat{\mathfrak{A}}$ if and only if $p(B)$ is a Borel subset of $\mathrm{Prim}(\mathfrak{A})$. Hence, if B is a Borel subset of $\hat{\mathfrak{A}}$, and if the relative Borel structure on B separates points, then p is one-to-one on B, and B and $p(B)$ are Borel isomorphic. But $\mathrm{Prim}(\mathfrak{A})$ is a standard Borel space by Theorem 2.4 of Effros [7]. Hence, $p(B)$, and therefore B, are standard Borel spaces.

Let q: $\mathrm{Irr}(\mathfrak{A}) \to \hat{\mathfrak{A}}$ be the natural continuous open mapping. As $\mathrm{Prim}(\mathfrak{A})$ is T_0 with a countable basis for its topology, each point of $\mathrm{Prim}(\mathfrak{A})$ is the intersection of a closed set and a G_δ. Hence, $q^{-1}(b) = (p \circ q)^{-1}(p(b))$ is a G_δ in $\mathrm{Irr}(\mathfrak{A})$ for all b in B. Each $q^{-1}(b)$ is also an F_σ by Lemma 2.7 and Lemma 4.1 of Effros [6]. Let R be the equivalence relation on $q^{-1}(B)$ given by point inverses under q. Each R-equivalence class is both a G_δ and an F_σ in $\mathrm{Irr}(\mathfrak{A})$. The R-saturation of a relatively open subset of $q^{-1}(B)$ is again relatively open, and therefore Borel, since $q|q^{-1}(B)$ is open onto B. Hence, Theorem 1 and Souslin's theorem show that $q^{-1}(B)/R$ and B are Borel isomorphic, and there is a cross section f: $B \to \mathrm{Irr}(\mathfrak{A})$. Q.E.D.

The following corollary has some applications. Consider the following setup. Let X be a standard Borel space, Y a Polish space, and R an equivalence relation on Y such that the R-saturation of open sets is Borel. R gives rise to an equivalence relation R' on $X \times Y$ by $R'(x,y) = \{x\} \times R(y)$.

Corollary 6. *Let B be a Borel subset of $X \times Y$ which is saturated with respect to R'. Suppose that each R'-equivalence class contained in B is, viewed as a subset of Y, both a G_δ and an F_σ. Then B/R' is standard, and there exists a Borel cross section f: $B/R' \to B$ for R'.*

Proof. There exists a Polish space Z and a one-to-one Borel mapping p: $Z \to X$. Let g: $(z,y) \to (p(z),y)$, $Z \times Y \to X \times Y$. Let $R'' = g^{-1}(R')$ and $B' = g^{-1}(B)$. Each R''-equivalence class contained in B' is both a G_δ and an F_σ since each R'-equivalence class in B is, viewed as a subset of Y, a G_δ and an F_σ, and since the vertical sections of $Z \times Y$ are closed. The R''-saturation of an open set in $Z \times Y$ is Borel. It suffices to prove this for open rectangles. Let $U \times V$ be open in $Z \times Y$, where U is open in Z and V is open in Y. But

$$R''(U \times V) = g^{-1}(R'(p(U) \times V)) = g^{-1}(p(U) \times R(V)),$$

which certainly is Borel in $Z \times Y$. Hence, by Theorem 1, B'/R'' is standard, and there exists a Borel cross section f': $B'/R'' \to B'$. Choose a sequence B'_n ($n \geq 1$) of R''-saturated Borel subsets of B' which separate the R''-equivalence classes. Then the $g(B'_n)$ ($n \geq 1$) are R'-saturated Borel subsets of B which separate the R'-equivalence classes. Hence, B/R' is countably separated. Furthermore, $g(f'(B'/R''))$ is a Borel transversal for the R'-equivalence classes of B. Let h: $g(f'(B'/R'')) \to B/R'$ be the natural one-to-one Borel mapping. Then B/R' is standard by Souslin's theorem, and $f = h^{-1}$: $B/R' \to B$ is a Borel cross section for R'. Q.E.D.

The following examples help to clarify the hypotheses of Theorem 1.

EXAMPLE 7. Note that Theorem 1 may fail if each R-equivalence class is only required to be an F_σ set, even if the R-saturation of each open set is open and B/R is metrizable. This follows from the fact that if A is an analytic nonborelian subset of J, the irrational numbers, then there is a Borel subset B of $J \times J$ such that the projection map restricted to B is open and projects B onto A. Also, each vertical section of B may be taken to be an F_σ subset of (see Taimanov [**11**]).

EXAMPLE 8. There is a Borel subset B of $J \times J$ such that each vertical section of B is an F_σ subset of J, the projection π onto the first axis, restricted to B, is open, $\pi(B) = J$, and yet there is no Borel cross section (in this case, there is no Borel uniformization). Recall that if E is a subset of $X \times Y$, then a uniformization of E is a subset F of E such that $E_x \neq \emptyset$ if and only if F_x consists of exactly one point, where $E_x = [y|(x, y) \text{ is in } E]$.

First, let M be a Borel subset of $J \times J$ such that $\pi(M) = J$, M has no Borel uniformization, and each vertical section of M is closed. The existence of such an M can be seen as follows. Let C_1 and C_2 be disjoint coanalytic subsets of J which are not Borel separable (see Sierpinski [**10**] for the existence of these C's). Let $A_1 = J - C_1$ and $A_2 = J - C_2$. A_1 and A_2 are analytic sets whose union is J. Let M_i be a closed subset of $J \times J$ which projects onto A_i ($i = 1, 2$). Let M be the Borel set which is the union of M_1 and M_2. If Γ were a Borel uniformization of M, then $D = \pi(\Gamma \cap (M_1 - M_2))$ would be a Borel subset of J which contains C_2 and has empty intersection with C_1. Thus, M has no Borel uniformization. This argument for the existence of M is due to D. Blackwell.

Identify J with N^N. Let $h_{n_1 \cdots n_k}$ be a homeomorphism of J onto $J(n_1, \ldots, n_k) = [z|z \text{ is in } J \text{ and } z_i = n_i \ (1 \leq i \leq k)]$, and let $T_{n_1 \cdots n_k}$: $(x, z) \to (x, h_{n_1 \cdots n_k}(z))$, $J \times J \to J \times J$. Let $B = \bigcup T_{n_1 \cdots n_k}(M)$. Then B is a Borel subset of $J \times J$, $\pi|B$ is open, $\pi(B) = J$, and each vertical section of B is an F_σ. If Γ were a Borel uniformization for B, then $C = \bigcup_n T_n^{-1}((\Gamma \cap T_n(M)) - \bigcup_{k<n} T_k(M))$ would be a Borel uniformization of M. Here k and n denote finite multi-indices and $<$ is the usual lexicographic order.

Suppose that B is a Borel subset of a Polish space X, R is an equivalence relation on B such that each equivalence class is a G_δ in X, and such that the saturation of relatively open sets is Borel. D. Miller has pointed out to the authors that Lemmas 3 and 4 may be altered slightly to prove that B/R is countably separated. Let V_m $(m \geqslant 1)$ be as in Lemma 4. We claim that the $R(V_m)$ $(m \geqslant 1)$ separate the R-equivalence classes. Let a and b be in X so that $R(a)$ and $R(b)$ are disjoint. If $R(b)$ is not contained in $\overline{R(a)}$, proceed as in Lemma 4. So suppose that $R(b)$ is contained in $\overline{R(a)}$. By a symmetric argument we may assume that $R(a)$ is contained in $\overline{R(b)}$. Thus, we may assume that $\overline{R(a)} = \overline{R(b)}$. But $R(a)$, being a G_δ, is comeager in $\overline{R(a)}$, and $R(b)$, being a G_δ, is comeager in $\overline{R(b)}$. Hence, $R(a) \cap R(b)$ is nonempty, a contradiction. The following questions remain. Is B/R standard? Even if B/R is standard, is there a cross section? The authors do not know the answers to these questions even if R is an open equivalence relation and B/R is metrizable. Note that if the last question has an affirmative answer, then there is a natural Borel cross section from $\mathrm{Prim}(\mathcal{Q}) \to \mathrm{Irr}(\mathcal{Q})$.

Bibliography

1. L. Auslander and B. Kostant, *Polarization and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354.

2. L. Auslander and C. C. Moore, *Unitary representations of solvable Lie groups*, Mem Amer. Math. Soc., no. 62 (1966).

3. N. Bourbaki, *General topology*. II, Hermann, Paris, 1966.

4. J. Dixmier, *Dual et quasi-dual d'une algèbre de Banach involutive*, Trans. Amer. Math. Soc. **104** (1962), 278–283.

5. ______, *Les C*-algebres et leurs representations*, Gauthier-Villars, Paris, 1969.

6. E. G. Effros, *Transformation groups and C*-algebras*, Ann. of Math. **81** (1965), 38–55.

7. ______, *A decomposition theory for representations of C*-algebras*, Trans. Amer. Math. Soc. **107** (1963), 83–106.

8. H. Hahn, *Reelle Funktionen*, Akad.-Verlag, Leipzig, 1932.

9. C. Kuratowski, *Topology*. I, Academic Press, New York, 1966.

10. W. Sierpinski, *Sur deux complémentaires analytiques non separables*, Fund. Math. **17** (1931), 296–297.

11. A. D. Taimanov, *On open images of Borel sets*, Mat. Sb. **37** (1955), 293–300.

12. G. W. Mackey, *Borel structure in groups and their duals*, Trans. Amer. Math. Soc. **85** (1957), 134–165.

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