BOUNDED SETS IN INDUCTIVE LIMITS

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Abstract. The Dieudonné-Schwartz theorem for bounded sets in strict inductive limits does not hold for general inductive limits. A set $B$ bounded in an inductive limit $E = \text{ind lim } E_n$ of locally convex spaces may not be contained in any $E_n$. If, however, each $E_n$ is closed in $E$, then $B$ is contained in some $E_n$, but may not be bounded there.

Let $E_1 \subset E_2 \subset \ldots$ be a sequence of locally convex spaces and $E = \text{ind lim } E_n$ their inductive limit (with respect to the identity maps $\text{id}: E_n \rightarrow E_{n+1}$). It is proved in [2, Chapter 2, §12], that a set $B \subset E$ is bounded iff it is contained and bounded in some $E_n$, provided that

(H-1) $E_n$ is closed in $E_{n+1}$ for each $n \in N$,
(H-2) for each $n \in N$, the topology of $E_n$ coincides with the topology induced on $E_n$ by $E_{n+1}$.

These two hypotheses imply [2, Chapter 2, §12]

(H-3) $E_n$ is closed in $E$ for each $n \in N$.

If we replace H-1 and H-2 by H-3, any set bounded in $E$ must still be contained in some $E_n$. But, as Example 1 shows, it may not be bounded in any $E_m$, $m > n$. Example 2 shows that, if we assume only H-1 instead of H-3, there may exist sets bounded in $E$ but not contained in any $E_n$.

Theorem. Let H-3 hold and $B$ be bounded in $E$. Then $B \subset E_n$ for some $n$.

Proof. Assume the contrary. Without loss of generality, we may assume that there exists a sequence $b_1, b_2, \ldots$ in $B$ such that $b_n \in E_n \setminus E_{n-1}$, $E_0 = \{0\}$, for all $n \in N$.

Since $b_1 \neq 0$, there exists a convex neighborhood $G_1$ of 0 in $E$ such that $b_1 \notin G_1 + G_1$. Put $V_1 = G_1 \cap E_1$. Then $V_1$ is a neighborhood of 0 in $E_1$ and $b_1 \notin V_1$. Suppose that in each $E_k$, $k = 1, 2, \ldots, n$ a neighborhood $V_k$ of 0 was chosen so that $V_1 \subset V_2 \subset \ldots \subset V_n$ and $b_m/m \notin (V_1 + \cdots + V_n)^E$, $m = 1, 2, n$. Put, for brevity, $W_k = (V_1 + \cdots + V_k)^E$. Since $E_n$ is closed in $E$, $W_n \subset E_n$ and there exists a convex neighborhood $G_{n+1}$ of 0 in $E$ such that

$$\frac{1}{m} b_m \notin W_n + G_{n+1} + G_{n+1} \quad \text{for all } m = 1, 2, \ldots, n + 1.$$
The set $V_{n+1} = G_{n+1} \cap E_{n+1}$ is a neighborhood of 0 in $E_{n+1}$ and

$$W_{n+1} = (V_1 + \cdots + V_{n+1})^E \subset V_1 + \cdots + V_{n+1}$$

$$+ G_{n+1} \subset V_1 + \cdots + V_n + G_{n+1} + G_{n+1} \subset W_n + G_{n+1} + G_{n+1}.$$  

This implies $b_m/m \notin W_{n+1}$ for $m = 1, 2, \ldots, n+1$.

The union of the nest $\{W_n; n \in \mathbb{N}\}$ is a neighborhood of 0 in $E$ which does not contain any $b_m/m$, $m \in \mathbb{N}$, and therefore does not absorb $B$.

The following notation is useful for our examples. For each $n \in \mathbb{N}$, $n$ will denote the set $(1, 2, \ldots, n)$ and $\mathbb{N} \setminus n$ the complement of $n$ in $\mathbb{N}$. Then, for instance, $A^n \times B^{\mathbb{N} \setminus n}$ denotes the set of all sequences of which the first $n$ terms are in $A$ and the remainder in $B$.

Example 1. Let $X$ be an infinite dimensional Banach space, $L$ its underlying linear space, and $Y$ the space $L$ endowed with its finest locally convex topology. Let, for each $n \in \mathbb{N}$, $E_n$ be the locally convex product $X^n \times Y^{\mathbb{N} \setminus n}$. As linear spaces, all the $E_n$ can be identified with $L^n$. The natural inductive limit $E = \text{ind lim } E_n$ can be as well. That $\text{H-3 holds, is trivial.}$

Claim. $E = X^n$. That the topology on $E$ is as fine as the product topology is evident. Let $W$ be any convex neighborhood of 0 in $E$. Then $W$ is a neighborhood of 0 in $E_1$ and so there exists some $n \in \mathbb{N}$ and $U$ a neighborhood of 0 in $Y^n$ such that

$$Q_1 = U^n \times L^{\mathbb{N} \setminus n} \subset W.$$  

Since $W$ is also a neighborhood of 0 in $E$, there exists neighborhoods of 0, $S$ in $X$ and $V$ in $Y$ and an integer $m \in \mathbb{N}$ such that

$$Q_2 = (2S)^n \times V^m \times L^{\mathbb{N} \setminus n + m} \subset W.$$  

Thus we have

$$S^n \times L^{\mathbb{N} \setminus n} \subset \frac{1}{2} Q_1 + \frac{1}{2} Q_2 \subset W.$$  

But $S^n \times L^{\mathbb{N} \setminus n}$ is a neighborhood of $X^n$, which proves our claim.

Finally we note on the one hand that, if $B$ is the unit ball in $X$, then $B^n$ is bounded in $E = X^n$, but, on the other hand, since $B$ is not finite dimensional and so unbounded in $Y$, $B^n$ is unbounded in each $E_n$.

Example 2. Let $X$, $Y$, and $B$ be as in Example 1, $Z$ a proper dense linear subspace of $X$ endowed with its finest locally convex topology, and $D$ be $B \cap Z$. For each $n \in \mathbb{N}$, let $E_n$ be the locally convex product $X^n \times Y \times Z^{\mathbb{N} \setminus n+1}$ and let $E$ be the inductive limit $\text{ind lim } E_n$ (with respect to the identity mapping id: $E_n \to E_{n+1}$). Since every linear subspace of $Z$ is closed, it is evident that each $E_n$ is closed in $E_{n+1}$: that $\text{H-1 holds.}$

Claim. $D^n$ is bounded in $E$. Let $G$ be any convex neighborhood of 0 in $E$. Then $G \cap E_1$ is a neighborhood of 0 in $E_1$ and so there exist neighborhoods of 0, $U$ in $Y$ and $A$ in $Z$ such that

$$Q_1 = U^2 \times A^n \times Z^{\mathbb{N} \setminus n+2} \subset G \text{ for some } n \in \mathbb{N}.$$  

Further $G \cap E_n$ is a neighborhood of 0 in $E_n$ and so there exist neighborhoods of 0, $S$ of $X$, $T$ of $Y$, and $V$ of $Z$ such that
\[ Q_2 = (2S)^{n+2} \times T \times V^m \times Z^{N \backslash n+m+3} \subset G \quad \text{for some } m \in N. \]

The set \( S^{n+2} \times Z^{N \backslash n+2} \) absorbs \( D^N \) and
\[ S^{n+2} \times Z^{N \backslash n+2} \subset 1/2 Q_1 + 1/2 Q_2 \subset G. \]

Hence \( G \) absorbs \( D \), which proves our claim.

If \( a \in B^N \cap E \), then \( a \in B^N \cap E_n \) for some \( n \in N \) and so \( a \) is a limit of a sequence in \( D^N \). Hence \( B^N \cap E \) is the closure of the bounded set \( D^N \), and so is bounded itself. But evidently \( B^N \cap E \) is not contained in any \( E_n \). Thus, though H-1 holds, it follows from the theorem of this paper that H-3 does not.

REFERENCES