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BOUNDED SETS IN INDUCTIVE LIMITS

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ABSTRACT. The Dieudonné-Schwartz theorem for bounded sets in strict inductive limits does not hold for general inductive limits. A set B bounded in an inductive limit $E = \text{ind lim } E_n$ of locally convex spaces may not be contained in any E_n. If, however, each E_n is closed in E, then B is contained in some E_n, but may not be bounded there.

Let $E_1 \subset E_2 \subset \ldots$ be a sequence of locally convex spaces and $E = \text{ind lim } E_n$ their inductive limit (with respect to the identity maps id: $E_n \to E_{n+1}$). It is proved in [2, Chapter 2, §12], that a set $B \subset E$ is bounded iff it is contained and bounded in some E_n, provided that

(H-1) E_n is closed in E_{n+1} for each $n \in N$,

(H-2) for each $n \in N$, the topology of E_n coincides with the topology induced on E_n by E_{n+1}.

These two hypotheses imply [2, Chapter 2, §12]

(H-3) E_n is closed in E for each $n \in N$.

If we replace H-1 and H-2 by H-3, any set bounded in E must still be contained in some E_n. But, as Example 1 shows, it may not be bounded in any E_m, $m \geqslant n$. Example 2 shows that, if we assume only H-1 instead of H-3, there may exist sets bounded in E but not contained in any E_n.

THEOREM. *Let* H-3 *hold and B be bounded in E. Then $B \subset E_n$ for some n.*

PROOF. Assume the contrary. Without loss of generality, we may assume that there exists a sequence $b_1, b_2, \ldots$ in B such that $b_n \in E_n \setminus E_{n-1}$, $E_0 = \{0\}$, for all $n \in N$.

Since $b_1 \neq 0$, there exists a convex neighborhood G_1 of 0 in E such that $b_1 \notin G_1 + G_1$. Put $V_1 = G_1 \cap E_1$. Then V_1 is a neighborhood of 0 in E_1 and $b_1 \notin V_1$. Suppose that in each E_k, $k = 1, 2, \ldots, n$ a neighborhood V_k of 0 was chosen so that $V_1 \subset V_2 \subset \cdots \subset V_n$ and $b_m/m \notin \overline{(V_1 + \cdots + V_n)}^E$, $m = 1, 2, \ldots, n$. Put, for brevity, $W_k = \overline{(V_1 + \cdots + V_k)}^E$. Since E_n is closed in E, $W_n \subset E_n$ and there exists a convex neighborhood G_{n+1} of 0 in E such that

$$\frac{1}{m} b_m \notin W_n + G_{n+1} + G_{n+1} \quad \text{for all } m = 1, 2, \ldots, n+1.$$

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The set $V_{n+1} = G_{n+1} \cap E_{n+1}$ is a neighborhood of 0 in E_{n+1} and

$$W_{n+1} = \left(\overline{V_1 + \cdots + V_{n+1}}\right)^E \subset V_1 + \cdots + V_{n+1} + G_{n+1} \subset V_1 + \cdots + V_n + G_{n+1} + G_{n+1} \subset W_n + G_{n+1} + G_{n+1}.$$

This implies $b_m/m \notin W_{n+1}$ for $m = 1, 2, \ldots, n+1$.

The union of the nest $\{W_n;\ n \in N\}$ is a neighborhood of 0 in E which does not contain any b_m/m, $m \in N$, and therefore does not absorb B.

The following notation is useful for our examples. For each $n \in N$, $\underline{n}$ will denote the set $\{1, 2, \ldots, n\}$ and $N \setminus \underline{n}$ the complement of $\underline{n}$ in N. Then, for instance, $A^{\underline{n}} \times B^{N \setminus \underline{n}}$ denotes the set of all sequences of which the first n terms are in A and the remainder in B.

EXAMPLE 1. Let X be an infinite dimensional Banach space, L its underlying linear space, and Y the space L endowed with its finest locally convex topology. Let, for each $n \in N$, E_n be the locally convex product $X^{\underline{n}} \times Y^{N \setminus \underline{n}}$. As linear spaces, all the E_n can be identified with L^N. The natural inductive limit $E = \text{ind lim } E_n$ can be as well. That H-3 holds, is trivial.

CLAIM. $E = X^N$. That the topology on E is as fine as the product topology is evident. Let W be any convex neighborhood of 0 in E. Then W is a neighborhood of 0 in E_1 and so there exists some $n \in N$ and U a neighborhood of 0 in Y such that

$$Q_1 \equiv U^{\underline{n}} \times L^{N \setminus \underline{n}} \subset W.$$

Since W is also a neighborhood of 0 in E_n, there exists neighborhoods of 0, S in X and V in Y and an integer $m \in N$ such that

$$Q_2 \equiv (2S)^{\underline{n}} \times V^{\underline{m}} \times L^{N \setminus \underline{n+m}} \subset W.$$

Thus we have

$$S^{\underline{n}} \times L^{N \setminus \underline{n}} \subset \tfrac{1}{2} Q_1 + \tfrac{1}{2} Q_2 \subset W.$$

But $S^{\underline{n}} \times L^{N \setminus \underline{n}}$ is a neighborhood of X^N, which proves our claim.

Finally we note on the one hand that, if B is the unit ball in X, then B^N is bounded in $E = X^N$, but, on the other hand, since B is not finite dimensional and so unbounded in Y, B^N is unbounded in each E_n.

EXAMPLE 2. Let X, Y, and B be as in Example 1, Z a proper dense linear subspace of X endowed with its finest locally convex topology, and D be $B \cap Z$. For each $n \in N$, let E_n be the locally convex product $X^{\underline{n}} \times Y \times Z^{N \setminus \underline{n+1}}$ and let E be the inductive limit $\text{ind lim } E_n$ (with respect to the identity mapping id: $E_n \to E_{n+1}$). Since every linear subspace of Z is closed, it is evident that each E_n is closed in E_{n+1}: that H-1 holds.

CLAIM. D^N is bounded in E. Let G be any convex neighborhood of 0 in E. Then $G \cap E_1$ is a neighborhood of 0 in E_1 and so there exist neighborhoods of 0, U in Y and A in Z such that

$$Q_1 = U^{\underline{2}} \times A^{\underline{n}} \times Z^{N \setminus \underline{n+2}} \subset G \quad \text{for some } n \in N.$$

Further $G \cap E_n$ is a neighborhood of 0 in E_n and so there exist neighborhoods of 0, S of X, T of Y, and V of Z such that

$$Q_2 = (2\,S)^{\underline{n+2}} \times T \times V^{\underline{m}} \times Z^{N\setminus\underline{n+m+3}} \subset G \quad \text{for some } m \in N.$$

The set $S^{\underline{n+2}} \times Z^{N\setminus\underline{n+2}}$ absorbs D^N and

$$S^{\underline{n+2}} \times Z^{N\setminus\underline{n+2}} \subset \tfrac{1}{2} Q_1 + \tfrac{1}{2} Q_2 \subset G.$$

Hence G absorbs D, which proves our claim.

If $a \in B^N \cap E$, then $a \in B^N \cap E_n$ for some $n \in N$ and so a is a limit of a sequence in D^N. Hence $B^N \cap E$ is the closure of the bounded set D^N, and so is bounded itself. But evidently $B^N \cap E$ is not contained in any E_n. Thus, though H-1 holds, it follows from the theorem of this paper that H-3 does not.

References

1. J. Dieudonné and L. Schwartz, *La dualité dans les espaces* (𝔉) *et* (𝔏𝔉), Ann. Inst. Fourier Grenoble **1** (1949), 61–101.

2. J. Horváth, *Topological vector spaces and distributions*, Vol. I, Addison-Wesley, Reading, Mass., 1966.

3. J. L. Kelley and I. Namioka, *Linear topological spaces*, Van Nostrand, Princeton, N.J., 1963.

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