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ORTHOGONAL DECOMPOSITION OF ISOMETRIES IN A BANACH SPACE

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ABSTRACT. In this paper the Wold decomposition theorem is proved for a class of isometries in smooth reflexive Banach spaces. The class in particular contains all isometries of $L^p(\mu)$ spaces for arbitrary measures μ.

It is well known that an isometry from a Hilbert space H into itself can be represented as the direct sum of two isometries, one being unitary (invertible) and the other being a shift. It is the purpose of this paper to investigate conditions which ensure that an isometry on a Banach space be represented in a similar fashion.

Let X be a Banach space. For $x, y \in X$ we shall say that x is *orthogonal* to y ($x \perp y$) if for each $\alpha \in \mathbf{C}$, $\|x\| \leq \|x + \alpha y\|$. We note that this is a nonsymmetric notion of orthogonality but that it is equivalent to the usual concept of orthogonality in Hilbert space. We write $M \perp N$ in case $M, N \subseteq X$ and $x \in M, y \in N \Rightarrow x \perp y$.

DEFINITION. A semi-inner-product (s.i.p.) on X is a function $[\cdot, \cdot]$ from $X \times X$ into $\mathbf{C}$ with the following properties:

(1) $[\cdot, y]$ is linear for each $y \in X$,

(2) $|[x, y]| \leq \|x\| \cdot \|y\|$,

(3) $[x, x] = \|x\|^2$ for each $x \in X$,

(4) $[x, \alpha y] = \bar{\alpha}[x, y]$ for each $x, y \in X$ and $\alpha \in \mathbf{C}$.

General facts concerning semi-inner-products may be found in [2], [5]. For $M, N \subseteq X$, we write $[M, N]$ for the collection of numbers of the form $[x, y]$ where $x \in M$ and $y \in N$. It is natural to define orthogonality in terms of the s.i.p., but a particular Banach space may have many s.i.p.'s consistent with the norm and the notions of orthogonality will be dependent on the s.i.p. However, we do have the following result:

THEOREM [1]. *Let X be a normed linear space and M and N be subspaces of X with $M \perp N$, then there is a s.i.p. $[\cdot, \cdot]$ such that $[N, M] = \{0\}$.*

In the present paper we shall be concerned with orthogonal complements and so we have the following two lemmas concerning such complements.

LEMMA 1. *Let $X = M \oplus N$ where M and N are subspaces of X with $M \perp N$. Then $N = \{x \in X: [x, M] = 0\}$ for some s.i.p. $[\cdot, \cdot]$.*

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PROOF. Choose a s.i.p. such that $[N, M] = 0$. Then if $[x, M] = 0$, express x as $x = m + n$ with $m \in M$, $n \in N$, then

$$0 = [x, m] = [m + n, m] = \|m\|^2,$$

so $x = n \in N$.

LEMMA 2. *Let M and N be closed subspaces of X with $M \perp N$, then $M \oplus N$ is closed.*

PROOF. Again choose a s.i.p. such that $[N, M] = 0$. Let $z_n = x_n + y_n$ where $x_n \in M$ and $y_n \in N$; if $z_n \to z \in X$, then

$$\begin{aligned}\|z_n - z_m\| \cdot \|x_n - x_m\| &\geq |[z_n - z_m, x_n - x_m]| \\ &= |[x_n + y_n - x_m - y_m, x_n - x_m]| \\ &= |[x_n - x_m, x_n - x_m] + [y_n - y_m, x_n - x_m]| \\ &= \|x_n - x_m\|^2.\end{aligned}$$

Thus $\|x_n - x_m\| \leq \|z_n - z_m\|$, so $\{x_n\}$ must be Cauchy and hence $x_n \to x$ for some $x \in M$. It follows that $y_n \to z - x$ and $z - x = y \in N$. Therefore $z_n \to x + y \in M \oplus N$ so $M \oplus N$ is closed.

In a smooth Banach space, the s.i.p. is unique so we may write $M^\perp$ to stand for $\{x | [x, M] = 0\}$.

LEMMA 3. *Let X be a smooth, reflexive Banach space and suppose that $\{M_k\}$ and $\{N_k\}$ are sequences of closed subspaces such that*

(1) $X = M_k \oplus N_k$,

(2) $N_k \perp M_k$,

(3) $N_k \subseteq N_{k-1}$ *and* $M_k \supseteq M_{k-1}$.

Let $N = \bigcap_{k=1}^\infty N_k$ *and* $M = \overline{[\bigcup_{k=1}^\infty M_k]}$; *then* $X = M \oplus N$ *and* $N \perp M$.

PROOF. Suppose $z = n_k + m_k$ where $n_k \in N_k$ and $m_k \in M_k$ for each natural number k; then

$$\|z\| = \|n_k + m_k\| \geq \|n_k\| \qquad \text{(by (2))}.$$

Thus $\{n_k\}$ is a bounded sequence in X and hence has a weakly convergent subsequence $\{n_{k_i}\}$. If n_{k_i} converges weakly to n, then $m_k = z - n_k$ converges weakly also, say to $m = z - n$. Now $n \in \bigcap_{k=1}^\infty N_k$ and $m \in \overline{[\bigcup_{k=1}^\infty M_k]}$. Thus $X = M + N$. If $n \in N$ and $m \in \bigcup M_k$, then $[m, n] = 0$. Since $[\cdot, n]$ is continuous, we have that if $n \in N$ and $m \in M$, then $[m, n] = 0$, so $N \perp M$. If $m \in N \cap M$, then $[m, m] = 0$ so $m = 0$; thus $X = M \oplus N$.

DEFINITION. Let V be an isometry on the normed linear space X. V is said to be *orthogonally complemented* provided that there exists a closed subspace M of X such that $X = M \oplus V(X)$ and $V(X) \perp M$.

We note that V is orthogonally complemented if and only if there exists a projection $P: X \to V(X)$ of norm 1.

In a Hilbert space, each isometry is orthogonally complemented. If

(Ω, Σ, μ) is a σ-finite measure space, then for $p \neq 2$ $(1 \leqslant p < \infty)$, each isometry V: $L^p(\Omega) \to L^p(\Omega)$ is of the form $Vf(x) = h(x)Tf(x)$ where T is a regular set isomorphism of Σ and $|h|^p = d\nu/d\mu$ for $\nu(A) = \mu(T^{-1}A)$ [4]. (Here Tf is defined by extending to all of $L^p(\Omega)$, the mapping $T(1_A) = 1_{TA}$.) Define g by

$$g(x) = \frac{1}{h(x)} 1_{\{h \neq 0\}}.$$

If $g \in L^p(\Omega)$, then V is orthogonally complemented. In particular, T is measure preserving if and only if $|h| = 1$ a.e.; so every classical shift for $L^p(\Omega)$ is orthogonally complemented (as is every isometry in l_p).[1] A similar condition can be given for isometries in certain Orlicz spaces [6].

Definition. An isometry V: $X \to X$ will be called a *unilateral shift* if there exists a subspace $L \subseteq X$ such that

(1) for $n \geqslant m$, $V^n(L) \perp V^m(L)$,

(2) $X = \bigoplus_{n=0}^{\infty} V^n(L)$.

The following theorem generalizes a result of Wold for isometries on Hilbert space.

Theorem. *Let V be an isometry on the smooth, reflexive Banach space X. If V is orthogonally complemented then there exist closed subspaces X_1 and X_2 such that*

(1) *X_1 and X_2 are invariant under V,*

(2) *$V|_{X_1}$ is unitary (surjective),*

(3) *$V|_{X_2}$ is a unilateral shift,*

(4) *$X = X_1 \oplus X_2$.*

Proof. Let $L = V(X)^{\perp}$, then by hypothesis, $X = V(X) \oplus L$. By [3] we know that $[Vx, Vy] = [x, y]$ so for $n \geqslant m$, $V^n(L) \perp V^m(L)$. Since X is smooth,

$$V^n(L) \perp \bigoplus_{k=0}^{n-1} V^k(L)$$

so by a previous lemma, the subspace $L_n = \bigoplus_{k=0}^{n} V^k(L)$ is closed for each n. Since $X = V(X) \oplus L$, then $V(X) = V^2(X) \oplus V(L)$; so

$$X = V^2(X) \oplus V(L) \oplus L = V^2(X) \oplus L_1 \quad \text{and} \quad V^2(X) \perp L_1.$$

In general, $X = V^n(X) \oplus L_{n-1}$ and $V^n(X) \perp L_{n-1}$. The sequence $\{V^n(X)\}$ decreases and the sequence $\{L_n\}$ increases so we set $X_1 = \bigcap_{n=0}^{\infty} V^n(X)$ and

$$X_2 = \overline{\bigcup_{n=0}^{\infty} L_n} = \bigoplus_{n=0}^{\infty} V^n(L).$$

Then by Lemma 3, $X = X_1 \oplus X_2$, and X_1 and X_2 are invariant under V. By construction, $V|_{X_2}$ is a shift and V is surjective from X_1 to X_1.

[1]The referee has kindly pointed out that a result of Ando [7] ensures that every isometry of $L^p(\Omega)$ $(1 \leqslant p < \infty, p \neq 2)$ is orthogonally complemented.

COROLLARY. *If V is an isometry on a smooth, reflexive Banach space which satisfies*:

(1) *V is orthogonally complemented,*

(2) $\bigcap_{n=0}^{\infty} V^n(X) = \{0\}$,

then V is a unilateral shift.

In [3] it is pointed out that for arbitrary Banach spaces it is unknown whether or not an eigenspace of an isometry has an invariant complement. However [3], this is known for invertible isometries. We have a partial solution to this problem given by:

COROLLARY. *Let V be an isometry in a smooth, reflexive Banach space. If V is orthogonally complemented then every eigenspace of V has an invariant complement.*

The proof follows from [3] and the previous theorem. Several questions are raised by these results. Two of these are as follows:

(1) In a Hilbert space a unilateral shift can be shown to be unitarily equivalent to an "actual" shift on $l^2(K)$ where $K = V(X)^{\perp}$. However, the proof does not seem to generalize in a natural fashion. Is there a similar result? If not, what operators satisfying the definitions of a unilateral shift are in some sense "natural" shifts?

(2) What isometries are not orthogonally complemented?

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