ORTHOGONAL DECOMPOSITION OF ISOMETRIES IN A BANACH SPACE

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Abstract. In this paper the Wold decomposition theorem is proved for a class of isometries in smooth reflexive Banach spaces. The class in particular contains all isometries of $L^p(\mu)$ spaces for arbitrary measures $\mu$.

It is well known that an isometry from a Hilbert space $H$ into itself can be represented as the direct sum of two isometries, one being unitary (invertible) and the other being a shift. It is the purpose of this paper to investigate conditions which ensure that an isometry on a Banach space be represented in a similar fashion.

Let $X$ be a Banach space. For $x, y \in X$ we shall say that $x$ is orthogonal to $y \ (x \perp y)$ if for each $\alpha \in \mathbb{C}$, $\|x\| < \|x + \alpha y\|$. We note that this is a nonsymmetric notion of orthogonality but that it is equivalent to the usual concept of orthogonality in Hilbert space. We write $M \perp N$ in case $M, N \subset X$ and $x \in M, y \in N \Rightarrow x \perp y$.

Definition. A semi-inner-product (s.i.p.) on $X$ is a function $\langle \cdot, \cdot \rangle$ from $X \times X$ into $\mathbb{C}$ with the following properties:

1. $\langle x, y \rangle$ is linear for each $y \in X$,
2. $\|\langle x, y \rangle\| \leq \|x\| \cdot \|y\|$, 
3. $\langle x, x \rangle = \|x\|^2$ for each $x \in X$,
4. $\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$ for each $x, y \in X$ and $\alpha \in \mathbb{C}$.

General facts concerning semi-inner-products may be found in [2], [5]. For $M, N \subset X$, we write $[M, N]$ for the collection of numbers of the form $\langle x, y \rangle$ where $x \in M$ and $y \in N$. It is natural to define orthogonality in terms of the s.i.p., but a particular Banach space may have many s.i.p.'s consistent with the norm and the notions of orthogonality will be dependent on the s.i.p. However, we do have the following result:

Theorem [1]. Let $X$ be a normed linear space and $M$ and $N$ be subspaces of $X$ with $M \perp N$, then there is a s.i.p. $\langle \cdot, \cdot \rangle$ such that $[N, M] = \{0\}$.

In the present paper we shall be concerned with orthogonal complements and so we have the following two lemmas concerning such complements.

Lemma 1. Let $X = M \oplus N$ where $M$ and $N$ are subspaces of $X$ with $M \perp N$. Then $N = \{x \in X: [x, M] = 0\}$ for some s.i.p. $\langle \cdot, \cdot \rangle$.
Proof. Choose a s.i.p. such that $[N, M] = 0$. Then if $[x, M] = 0$, express $x$ as $x = m + n$ with $m \in M$, $n \in N$, then

$$0 = [x, m] = [m + n, m] = \|m\|^2,$$

so $x = n \in N$.

**Lemma 2.** Let $M$ and $N$ be closed subspaces of $X$ with $N \perp M$, then $M \oplus N$ is closed.

Proof. Again choose a s.i.p. such that $[N, M] = 0$. Let $z_n = x_n + y_n$ where $x_n \in M$ and $y_n \in N$; if $z_n \to z \in X$, then

$$\|z_n - z_m\| \geq \|[x_n - x_m, x_n - x_m]\| = ||x_n + y_n - x_m - y_m, x_n - x_m|| = ||x_n - x_m||^2.$$

Thus $\|x_n - x_m\| \leq \|z_n - z_m\|$, so $\{x_n\}$ must be Cauchy and hence $x_n \to x$ for some $x \in M$. It follows that $y_n \to z - x$ and $z - x = y \in N$. Therefore $z_n \to z + y \in M \oplus N$ so $M \oplus N$ is closed.

In a smooth Banach space, the s.i.p. is unique so we may write $M \perp$ to stand for $\{x \mid [x, M] = 0\}$.

**Lemma 3.** Let $X$ be a smooth, reflexive Banach space and suppose that $\{M_k\}$ and $\{N_k\}$ are sequences of closed subspaces such that

1. $X = M_k \oplus N_k$,
2. $N_k \perp M_k$,
3. $N_k \subseteq N_{k-1}$ and $M_k \supseteq M_{k-1}$.

Let $N = \bigcap_{k=1}^{\infty} N_k$ and $M = \bigcup_{k=1}^{\infty} M_k$; then $X = M \oplus N$ and $N \perp M$.

Proof. Suppose $z = n_k + m_k$ where $n_k \in N_k$ and $m_k \in M_k$ for each natural number $k$; then

$$\|z\| = \|n_k + m_k\| \geq \|n_k\| \quad \text{(by (2))}.$$

Thus $\{n_k\}$ is a bounded sequence in $X$ and hence has a weakly convergent subsequence $\{n_{k_n}\}$. If $n_{k_n}$ converges weakly to $n$, then $m_k = z - n_k$ converges weakly also, say to $m = z - n$. Now $n \in \bigcap_{k=1}^{\infty} N_k$ and $m \in \bigcup_{k=1}^{\infty} M_k$.

Thus $X = M + N$. If $n \in N$ and $m \in M_k$, then $[m, n] = 0$. Since $[\cdot, \cdot]$ is continuous, we have that if $n \in N$ and $m \in M$, then $[m, n] = 0$, so $N \perp M$. If $m \in N \cap M$, then $[m, m] = 0$ so $m = 0$; thus $X = M \oplus N$.

**Definition.** Let $V$ be an isometry on the normed linear space $X$. $V$ is said to be orthogonally complemented provided that there exists a closed subspace $M$ of $X$ such that $X = M \oplus V(X)$ and $V(X) \perp M$.

We note that $V$ is orthogonally complemented if and only if there exists a projection $P: X \to V(X)$ of norm 1.

In a Hilbert space, each isometry is orthogonally complemented. If
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$(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space, then for $p \neq 2$ ($1 < p < \infty$), each isometry $V: L^p(\Omega) \to L^p(\Omega)$ is of the form $Vf(x) = h(x)Tf(x)$ where $T$ is a regular set isomorphism of $\Sigma$ and $|h|^p = dv/du$ for $\nu(A) = \mu(T^{-1}A)$ \[4\]. (Here $Tf$ is defined by extending to all of $L^p(\Omega)$, the mapping $T(1_A) = 1_{TA}$.)

Define $g$ by

$$g(x) = \frac{1}{h(x)} 1_{(h \neq 0)}.$$

If $g \in L^p(\Omega)$, then $V$ is orthogonally complemented. In particular, $T$ is measure preserving if and only if $|h| = 1$ a.e.; so every classical shift for $L^p(\Omega)$ is orthogonally complemented (as is every isometry in $L_p$).\footnote{The referee has kindly pointed out that a result of Ando \[7\] ensures that every isometry of $L^p(\Omega)$ $(1 < p < \infty, p \neq 2)$ is orthogonally complemented.}

A similar condition can be given for isometries in certain Orlicz spaces \[6\].

**Definition.** An isometry $V: X \to X$ will be called a **unilateral shift** if there exists a subspace $L \subseteq X$ such that

1. for $n > m$, $V^n(L) \perp V^m(L)$,
2. $X = \bigoplus_{n=0}^{\infty} V^n(L)$.

The following theorem generalizes a result of Wold for isometries on Hilbert space.

**Theorem.** Let $V$ be an isometry on the smooth, reflexive Banach space $X$. If $V$ is orthogonally complemented then there exist closed subspaces $X_1$ and $X_2$ such that

1. $X_1$ and $X_2$ are invariant under $V$,
2. $V|_{X_1}$ is unitary (surjective),
3. $V|_{X_2}$ is a unilateral shift,
4. $X = X_1 \oplus X_2$.

**Proof.** Let $L = V(X)^\perp$, then by hypothesis, $X = V(X) \oplus L$. By \[3\] we know that $[Vx, Vy] = [x, y]$ so for $n > m$, $V^n(L) \perp V^m(L)$. Since $X$ is smooth,

$$V^n(L) \perp \bigoplus_{k=0}^{n-1} V^k(L),$$

so by a previous lemma, the subspace $L_n = \bigoplus_{k=0}^{n-1} V^k(L)$ is closed for each $n$. Since $X = V(X) \oplus L$, then $V(X) = V^2(X) \oplus V(L)$; so

$$X = V^2(X) \oplus V(L) \oplus L = V^2(X) \oplus L_1 \quad \text{and} \quad V^2(X) \perp L_1.$$\[5\]

In general, $X = V^n(X) \oplus L_{n-1}$ and $V^n(X) \perp L_{n-1}$. The sequence $\{V^n(X)\}$ decreases and the sequence $\{L_n\}$ increases so we set $X_1 = \bigcap_{n=0}^{\infty} V^n(X)$ and

$$X_2 = \bigcup_{n=0}^{\infty} L_n = \bigoplus_{n=0}^{\infty} V^n(L).$$

Then by Lemma 3, $X = X_1 \oplus X_2$, and $X_1$ and $X_2$ are invariant under $V$. By construction, $V|_{X_2}$ is a shift and $V$ is surjective from $X_1$ to $X_1$.\[6\]
Corollary. If $V$ is an isometry on a smooth, reflexive Banach space which satisfies:

1. $V$ is orthogonally complemented,
2. $\bigcap_{n=0}^{\infty} V^n(X) = \{0\}$,

then $V$ is a unilateral shift.

In [3] it is pointed out that for arbitrary Banach spaces it is unknown whether or not an eigenspace of an isometry has an invariant complement. However [3], this is known for invertible isometries. We have a partial solution to this problem given by:

Corollary. Let $V$ be an isometry in a smooth, reflexive Banach space. If $V$ is orthogonally complemented then every eigenspace of $V$ has an invariant complement.

The proof follows from [3] and the previous theorem. Several questions are raised by these results. Two of these are as follows:

1. In a Hilbert space a unilateral shift can be shown to be unitarily equivalent to an "actual" shift on $l^2(K)$ where $K = V(X)$. However, the proof does not seem to generalize in a natural fashion. Is there a similar result? If not, what operators satisfying the definitions of a unilateral shift are in some sense "natural" shifts?

2. What isometries are not orthogonally complemented?

References