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A NOTE ON KILLING TORSION OF MANIFOLDS BY SURGERY

STAVROS PAPASTAVRIDIS

ABSTRACT. In this note we prove that every manifold of dimension different than 3, oriented in a bundle theory (for most bundle theories), is cobordant to a manifold which contains not more torsion than the classifying space of the bundle theory.

We will give here a generalization (with simpler proof) of a theorem of R. Stong which says that every closed oriented manifold is cobordant, in the oriented sense, to an oriented manifold whose homology contains no odd torsion [3].

DEFINITION 1. If G is an abelian group, then Tor G is the subgroup of G which consists of torsion elements. Let $G = (G_0, G_1, \dots)$ and $H = (H_0, H_1, \dots)$ be graded abelian groups, then we say that "the torsion of G is contained in the torsion of H", if and only if for every i there is a j such that Tor G_i is isomorphic with a subgroup of Tor H_j.

Let f_r: $X_r \to BO(r)$ be a sequence of fibrations with maps g_r: $X_r \to X_{r+1}$ such that the usual diagram commutes. For such a situation R. Lashof defines the concept of X-structure on manifolds (see [2]) and proves a Thom-isomorphism for the bordism groups of such manifolds. It is well known that many of the usual classes of manifolds may be described in terms of X-structures, e.g. SO, U, Spin, etc. We assume that for r big enough X_r has a finite number of cells in each dimension, and that the map $(g_r)_*$: $H_*(X_r; Z) \to H_*(X_{r+1}; Z)$ "stabilizes" (namely for given n, it is an isomorphism up to dimension n, provided that r is big enough). Finally we assume that $f_r^*(w_1) = 0$, for r very big. Let $H_*(X) = \text{ind lim } H_*(X_r; Z)$.

THEOREM 2. *Every even dimensional, closed, X-manifold is X-cobordant to a manifold whose torsion is contained in the torsion of $H_*(X)$.*

THEOREM 3. *If X_r is simply connected for r big enough, then every* $(4k + 1)$-*dimensional X-manifold is X-cobordant to a manifold whose torsion is contained in the torsion of $H_*(X)$.*

THEOREM 4. *If X_r is simply connected for r big enough, and $H_{2k-1}(X)$ contains no p-torsion for certain prime numbers p (we take $k > 1$), then every*

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$(4k-1)$-dimensional X-manifold is X-cobordant to a manifold whose torsion is contained in the torsion of $H_*(X)$ in dimensions different than $(2k-1)$, and in dimension $(2k-1)$ contains no p-torsion for the same prime numbers.

PROOF OF THEOREMS 2,3. Let $\nu: M \to X_r$ be the normal map induced by the X-structure of M. We take r very big. Let m be the dimension of M.

By surgery (see Theorem IV.1.13 of [**1**]) we can make the map ν an isomorphism in homology up to dimension $[m/2]-1$, provided $m \geqslant 4$.

If X_r is simply connected and m is of the form $(4k+1)$ (we take k positive), then we can squeeze in an extra dimension to make ν a monomorphism in homology up to dimension $[m/2]$. This is proved exactly like Theorem IV.2.1 of [**1**].

And the proof follows from Poincaré duality and the fact that from the universal coefficient theorem, for any topological space Y, we have Tor $H^i(Y; Z) =$ Tor $H_{i-1}(Y; Z)$. The cases of 1, 2-dimensional manifolds offer no difficulty because such manifolds are torsion free.

PROOF OF THEOREM 4. As before, let $\nu: M \to X_r$ be the normal map induced by the X-structure of the $(4k-1)$-dimensional manifold M. Let p be a prime number such that $H_{2k-1}(X)$ contains no p-torsion.

By Theorem IV.2.1 of [**1**] (see particularly Proposition IV.3.12), we can surger M to a manifold M_1, so that the map $\nu_1: M_1 \to X_r$ is an isomorphism in homology up to dimension $(2k-2)$, and the map

$$(\nu_1)_*: H_{2k-1}(M_1; Z_p) \to H_{2k-1}(X; Z_p)$$

is a monomorphism. Consider the following commutative diagram

$$\begin{array}{ccccc} 0 \to & H_{2k-1}(M_1) \otimes Z_p & \to & H_{2k-1}(M_1; Z_p) \\ & \downarrow & & \downarrow \\ 0 \to & H_{2k-1}(X) \otimes Z_p & \to & H_{2k-1}(X; Z_p) \end{array}$$

where the horizontal maps are monomorphisms by the universal coefficient theorem, and the second vertical map is a monomorphism as explained before. But this implies that the first vertical map is a monomorphism, and the fact that $H_{2k-1}(X)$ contains no p-torsion implies that the same is true for $H_{2k-1}(M_1)$. That ends the proof.

I do not know what happens in dimension 3.

I am grateful to the editor for pointing out that the argument on p. 107 of Browder's book [**1**], does not carry in the $(4k-1)$-dimensional case.

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MATHEMATICAL INSTITUTE, UNIVERSITY OF ATHENS, SOLONOS 57, ATHENS 143, GREECE

Current address: Department of Mathematics, University of Crete, Iraklion, Crete, Greece