ON A RESULT OF OSBORN

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Abstract. The structure of certain semiprimitive rings with involution \(*\) is determined by imposing conditions on the set of \(*\)-symmetric elements and limiting the number of orthogonal \(*\)-symmetric idempotents.

An associative ring \(R\) with unit such that \(1/2 \in R\) satisfies condition \(C(n)\) if (i) \(R\) has an involution \(*\) such that each \(*\)-symmetric element \(s\) of \(R\) is nilpotent or some (right) multiple of \(s\) is a nonzero \(*\)-symmetric idempotent and (ii) \(R\) has a set of \(n\) nonzero, pairwise orthogonal, \(*\)-symmetric idempotents whose sum is one and if \(\{e_i\}_{i=1}^m\) is any set of such idempotents whose sum is one, then \(m \leq n\). We will determine the structure of all rings with condition \(C(n)\).

A ring satisfying \(C(1)\) has exactly one nonzero \(*\)-symmetric idempotent, the one of that ring. Hence each \(*\)-symmetric element is either nilpotent or invertible. Osborn [4] catalogued these rings and showed that a semiprimitive ring has \(C(1)\) iff \(R\) is one of

(i) a division ring,

(ii) a direct sum of two anti-isomorphic division rings with involution interchanging the summands,

(iii) the \(2 \times 2\) matrices over a field with the involution fixing only the scalar matrices.

First we reduce to the case where \(R\) is semiprimitive. Then we collect some of the facts to be used for our main result (Theorem 4). Since the results of Lemma 1 and Lemma 2 are well known, their proofs are omitted.

Lemma 1. If each \(*\)-symmetric element \(s\) of \(R\) is either nilpotent or some (right) multiple of \(s\) is a nonzero \(*\)-symmetric idempotent, then the Jacobson radical of \(R\), \(\mathfrak{J}(R)\), is a \(*\)-invariant ideal in which every \(*\)-symmetric element is nilpotent.

Lemma 2. If each \(*\)-symmetric element \(s\) of \(R\) is either nilpotent or some (right) multiple of \(s\) is a nonzero \(*\)-symmetric idempotent, and \(\bar{u} \in R/\mathfrak{J}(R)\) is a symmetric element under the induced involution \(*'\), then \(\bar{u}\) is either nilpotent or some (right) multiple of \(\bar{u}\) is a nonzero \(*'\)-symmetric idempotent.

Remark 1. Suppose \(R\) is a ring and for some \(a \in R\), \(a^2 - a\) is nilpotent. Then either \(a\) is nilpotent or for some polynomial \(q(x)\) with integer
coefficients $e = aq(a)$ is a nonzero idempotent. Furthermore, the sum of the coefficients of $q(a)$ is one. This is Lemma 1.3.2 of Herstein [1].

**Lemma 3.** Suppose each $\dagger$-symmetric element $s$ of $R$ is either nilpotent or some (right) multiple of $s$ is a nonzero $\dagger$-symmetric idempotent and let $\tilde{u}$ be a $\dagger$-symmetric idempotent of $R/J(R)$. Then $\tilde{u}$ can be lifted to a $\dagger$-symmetric idempotent of $R$.

**Proof.** We may assume that $u^* = u$. Since $u^2 - u$ is a $\dagger$-symmetric element in $J(R)$, it is nilpotent. Remark 1 tells us that there is a $\dagger$-symmetric idempotent $e = uq(u)$. Since the sum of the coefficients of $q(u)$ is one, $\tilde{e} = \tilde{u}$.

Even more can be said about lifting $\dagger$-symmetric idempotents.

**Theorem 1.** Suppose each $\ddagger$-symmetric element $s$ of $R$ is either nilpotent or some (right) multiple of $s$ is a nonzero $\dagger$-symmetric idempotent. Then if $\{\tilde{u}_i\}_{i=1}^n$ is a set of $\dagger$-symmetric, pairwise orthogonal idempotents in $R/J(R)$, then there exists a set $\{e_i\}_{i=1}^n$ of $\ddagger$-symmetric, pairwise orthogonal idempotents in $R$ with $\tilde{e}_i = \tilde{u}_i$.

**Proof.** By Lemma 3, $\tilde{u}_1$ can be lifted to the $\dagger$-symmetric idempotent $e_1$ and $\tilde{u}_2$ can be lifted to an idempotent $f$. Hence $e_1f$ and $fe_1$ are in $J(R)$. In particular, $1 - fe_1$ has an inverse in $R$ and we may form

$$f' = (1 - fe_1)^{-1}f(1 - fe_1).$$

This is an idempotent of $R$ and $f'e_1 = 0$. Multiplying by $1 - fe_1$ on the left, we see that $f' - f \in J(R)$.

Now put $h = f' - e_1f'$. Then $e_1h = 0 = he_1, h - u_2 \in J(R)$ and $h^2 = h$. Since we can assume $u_2^* = u_2, h^* - u_2 \in J(R)$. Thus $hh^* - u_2 \in J(R)$ and $e_1(hh^*) = (hh^*)e_1 = 0$. Now $hh^*$ is not nilpotent, but $(hh^*)^2 - (hh^*) \in J(R)$, so by Remark 1 some polynomial in $hh^*$ is a $\dagger$-symmetric idempotent $e_2$ and $e_2 - hh^* \in J(R)$. Thus $e_2 - u_2 \in J(R)$ and $e_1e_2 = e_2e_1 = 0$ since $e_2 = hh^*q(hh^*)$.

Suppose the first $n - 1$ elements of $\{\tilde{u}_i\}_{i=1}^n$ have been lifted to a set of $\{e_i\}_{i=1}^{n-1}$ $\ddagger$-symmetric, pairwise orthogonal idempotents. By Lemma 3, we can lift $\tilde{u}_n$ to an idempotent $f_n$. Then $(\Sigma_{i=1}^{n-1}e_i)f_n$ and $f_n\Sigma_{i=1}^{n-1}e_i$ are in $J(R)$. In particular, $1 - f_n\Sigma_{i=1}^{n-1}e_i$ has an inverse in $R$. Set

$$f'_n = \left(1 - f_n\sum_{i=1}^{n-1}e_i\right)^{-1}f_n\left(1 - f_n\sum_{i=1}^{n-1}e_i\right).$$

Put $h_n = f'_n - (\Sigma_{i=1}^{n-1}e_i)f'_n$. As above we can construct a polynomial $e_n$ in $h_nh_n^*$ such that $e_n$ is a $\ddagger$-symmetric idempotent, $e_n - u_n \in J(R)$ and $\{e_i\}_{i=1}^n$ is a set of pairwise orthogonal $\ddagger$-symmetric idempotents.

Hence $R$ has at least $m$ $\ddagger$-symmetric, pairwise orthogonal idempotents.

The results of Lemma 1 through Theorem 1 are summarized in

**Theorem 2.** If $R$ has $C(n)$, then so does $R/J(R)$. 

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Proof. \( R/J(R) \) has at least \( n \) \( \ast \)-symmetric pairwise orthogonal idempotents that it inherits from \( R \). By Theorem 1, it can have no more.

Remark 2. If a semiprimitive ring has \( C(1) \), then there is an idempotent \( f \in R \) such that \( fRf \) is a division ring (equivalently \( fR \) is a minimal right ideal \( R \)).

Remark 3. If \( R \) is a ring with involution whose symmetric elements are all nilpotent, then \( R \) is a radical ring. This is Lemma 3 of Osborn [4].

Remark 4. If \( e \) is a nonzero idempotent of a ring \( R \), then in the two-sided Peirce decomposition of \( R \) relative to \( e \),

\[
R = eRe + eR(1 - e) + (1 - e)Re + (1 - e)R(1 - e)
\]

the ring \( eRe \) is radical free if \( R \) is radical free.

Remark 5. Jacobson and Rickart [3] define a canonical involution \( \ast \) in \( R_n \) (the ring of \( n \times n \) matrices over \( R \)) as an involution in \( R_n \) such that \( e_{ii}^\ast = e_{ii} \), \( i = 1, 2, \ldots, n \). If \( \ast \) is a canonical involution in \( R_n \), then there is an involution \( r \to \tilde{r} \) in \( R \) and invertible elements \( \delta_i \in R \) such that \( \tilde{r} = r \) and

\[
(\sum \alpha_{ij} e_{ij})^\ast = \sum \delta_j^{-1} \tilde{\alpha}_{ij} \delta_i e_{ij}.
\]

Let \( R \) be a simple ring with the minimum condition for right ideals and involution. Then \( R = \Delta_n, \Delta \) a division ring. The involution is canonical except when \( \Delta \) is a field \( \Phi, n = 2m \) and \( x \to q^{-1}x'q \), where \( x' \) denotes the transpose of \( x \), \( q \) is the diagonal \( m \times m \) matrix over \( \Phi_2 \) with nonzero entries \( \sigma = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \).

In this case we can regard \( R \) as \( S_m \), where \( S = \Phi_2 \). If we introduce the involution \( \alpha \to \tilde{\alpha} = \sigma^{-1}a' \sigma \) in \( \Phi_2 \), then the given involution in \( R \) is canonical with all \( \delta_i = 1 \).

Lemma 4. Let \( R \) be a ring with \( C(n) \) and \( \{e_i\}_{i=1}^{n} \) a collection of pairwise orthogonal nonzero \( \ast \)-symmetric idempotents such that \( \sum_{i=1}^{n} e_i = 1 \). Then the ring \( e_iR_i e_i \) has \( C(1) \).

Proof. Look at the ring \( e_iR_i e_i \) in the two-sided Peirce decomposition of \( R \) relative to \( e_i \). This is a \( \ast \)-invariant subring of \( R \) and each \( \ast \)-element of \( e_iR_i e_i \) is either invertible in \( e_iR_i e_i \) or nilpotent. That is, \( e_iR_i e_i \) has \( C(1) \).

Lemma 5. Let \( R \) be a semiprimitive ring with \( C(n) \). Then \( R \) has the minimum condition on the right ideals.

Proof. \( R = \bigoplus \Sigma_{i=1}^{n} e_iR, \) a direct sum of right \( R \) modules. But since \( e_iR_i \) has \( C(1) \) there is an \( f_j \in e_iR_i \) such that \( f_j^2 = f_j, e_i f_j = f_j e_i \) and each of \( f_jRf_j \) and \( (e_i - f_j)R(e_i - f_j) \) is a division ring. Hence

\[
R = \bigoplus \sum_{i=1}^{n} \left[ f_j e_i R + (e_i - f_j) R \right]
\]
a finite direct sum of minimal right ideals.

Remark 6. It is well know that if a ring \( R \) with involution \( \ast \) has no proper \( \ast \)-invariant ideals, then \( R \) is either a simple ring or \( R \) is a direct sum of two simple rings with the involution interchanging the summands.
Remark 7. Let $A$ be an associative algebra over a field $\Phi$ and suppose that $a$ is a non-nilpotent non-invertible algebraic element of $A$. Then there is an idempotent $e = a^k P(a)$, where $P(a)$ lies in the subalgebra formally generated by $1$ and $a$, such that $a^k e = a^k$ for some integer $k > 1$. (See, for example, the proof in Jacobson [2, p. 210, Proposition 1].)

Theorem 3. A simple ring $R$ has $C(n)$ iff it is a ring of $n \times n$ matrices over (i) a division ring or over (ii) the $2 \times 2$ matrices over a field.

Proof. By Lemma 5, $R$ has minimal right ideals. By Remark 5, $R$ is a matrix ring with canonical involution and hence one of the two rings listed.

The main result can now be stated.

Theorem 4. A ring $R$ has property $C(n)$ iff

(i) $J(R)$ is a $\star$-invariant ideal in which every $\star$-symmetric element is nilpotent and

(ii) $R/J(R)$ is the direct sum of semiprimitive rings $R_n$ where each $R_n$ has property $C(n_i)$, $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} n_i = n$.

Proof. (i) This is established in Lemma 1.

(ii) Theorem 2 shows that $R/J(R)$ has $C(n)$. By Lemma 5, $R/J(R)$ has the minimum condition on right ideals. Hence $R/J(R)$ is the ring direct sum of matrices $M_i$. Since $R/J(R)$ has an involution $\star$, each $M_i$ is either fixed under $\star$ and hence satisfies $C(n_i)$ for some $n_i$ or $M_i$ is mapped onto $M_i^\star$ and then $(M_i \oplus M_i^\star)$ is fixed under $\star$ and satisfies $C(n_i)$ for some $n_i$ (Remark 6).

That any collection of such semiprimitive rings put together in this fashion has the stated property is evident.

Corollary 1. Let $R$ be an associative algebra with $1$ over the field $\Phi$ not of characteristic $2$. If $R$ has an involution $\star$ such that for each $\star$-symmetric element $s$ of $R$ there is a $\lambda(s) = \lambda^\star(s) \in \Phi$ and $s^2 - s \lambda(s)$ is either nilpotent or invertible, then $R/J(R)$ has $C(1)$ or $C(2)$.

Proof. Pass to $R/J(R)$ as in Lemma 2 and note that each $\star'$-symmetric element $s$ of $R/J(R)$ is either (i) invertible, (ii) nilpotent, or (iii) determines a nonzero $\star'$-symmetric idempotent $e = s^k P_s(s)$. $R/J(R)$ can have at most two nonzero pairwise orthogonal $\star'$-symmetric idempotents whose sum is one. Suppose otherwise. Let $\{e_i\}_{i=1}^{3}$ be a set of $\star'$-symmetric idempotents whose sum is one. Then $e_1 + 2e_2$ is a $\star'$-symmetric element and $[(e_1 + 2e_2)^2 - (e_1 + 2e_2)\lambda]^k = 0$ for some positive integer $k$. This last expression cannot be solved for $\lambda$.

Corollary 2. Let $A$ be an associative algebraic algebra with $1$ over the field $\Phi$ not of characteristic $2$. Let $A$ have an involution $\star$ and a set of $n$ nonzero, pairwise orthogonal, $\star$-symmetric idempotents whose sum is one. Then $A$ has $C(n)$ iff for each set $\{e_i\}_{i=1}^{m}$ of such idempotents whose sum is one, $m < n$.

Proof. We only need to show that each $\star$-symmetric element $s$ of $A$ is nilpotent or some (right) multiple of $s$ is a nonzero $\star$-symmetric idempotent. But this is true by Remark 7.
COROLLARY 3. Let $A$ be an associative algebraic algebra with 1 over the field $\Phi$ not of characteristic 2. If each nonnilpotent noninvertible $\ast$-symmetric element is algebraic over $\Phi$, then any finite set of pairwise orthogonal $\ast'$-symmetric idempotents in $A/J(A)$ can be lifted to an orthogonal set of $\ast$-symmetric idempotents of $A$.

PROOF. By Remark 7, each $\ast$-symmetric element $s$ is either nilpotent or some (right) multiple of $s$ is a nonzero $\ast$-symmetric idempotent. Then argue as in Theorem 1.

REFERENCES


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