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HEIGHT OF EXCEPTIONAL CHARACTER DEGREES[1]

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ABSTRACT. Suppose a finite group G has an abelian Sylow p-group P which is a T.I. set and whose normalizer is a Frobenius group with kernel P. It is shown that the degree of an exceptional character of G is not divisible by p. That is, exceptional characters have height 0 in the principal p-block of G.

We assume

(*) *P is a Sylow p-group of the finite group G, P is an Abelian T.I. set, and $N = N_G(P)$ is a Frobenius group with Frobenius kernel P.*

From the Brauer-Suzuki theory of exceptional characters we find the following facts relating characters of G and N when G has at least two classes of p-elements. Let $\lambda_1, \dots, \lambda_t$ be the exceptional characters of N. There are an integer c and a sign $\delta = \pm 1$ and exceptional characters $\Lambda_1, \dots, \Lambda_t$ of G such that $\Lambda_i(g) = \delta\lambda_i(g) + c$ for all $i = 1, \dots, t$ and all $g \in P^{\#}$. All other irreducible characters of G are constant on $P^{\#}$. We exploit this last fact to show

THEOREM. *Suppose (*) holds and G has at least two classes of p-elements. Then $p \nmid \Lambda_i(1)$ for any i.*

Recently the author showed that $c = 0$ when G has at least three classes of p-elements [1]. Since $\Lambda_i(1) \equiv \delta s + c \pmod{|P|}$, where $s = |N : P|$, the Theorem follows trivially in this case. Thus, the only new part of the Theorem is the case for which G has exactly two classes of p-elements.

We now begin the proof. The values of the Λ_i on p-regular elements of G are independent of i, so we will drop the subscript i for such elements. For any $x, y, z \in G$ we define class multiplication constants $\gamma(x, y, z)$ in the usual way. Choose $a, b, c \in P^{\#}$ and x a p-regular element of G. The usual character formula for class multiplication constants yields

$$\gamma(a, b, x) - \gamma(a, c, x) = \frac{|G|}{|P|^2} \frac{\Lambda(x)}{\Lambda(1)} \sum_i (\Lambda_i(a)\Lambda_i(b) - \Lambda_i(a)\Lambda_i(c))$$

$$= \frac{|G|}{|P|} \frac{\Lambda(x)}{\Lambda(1)} (\delta_{ab} - \delta_{ac})$$

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by an easy calculation involving column orthogonality. Here δ_{gh} is defined for any $g, h \in P^{\#}$ via

$$\delta_{gh} = \begin{cases} 1 & \text{if } g \sim h^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Since G has at least two classes of p-elements, we can choose $a, b, c \in P^{\#}$ so that $(\delta_{ab} - \delta_{ac}) = 1$. Thus

$$\gamma(a, b, x) - \gamma(a, c, x) = \frac{|G|}{|P|} \frac{\Lambda(x)}{\Lambda(1)}$$

is an integer.

Now suppose by way of contradiction that $p|\Lambda(1)$. Then also $p|\Lambda(x)$ for any p-regular element x by the above calculation. We now calculate $\|\Lambda_i\|^2$. This gives

$$1 = \|\Lambda_i\|^2 = \frac{\Lambda(1)^2}{|G|} + \sum{}' \frac{\Lambda_i(g)\overline{\Lambda_i(g)}}{|P|} + p^2R$$

where the sum Σ' is over a set of class representatives g and R is some p-local rational number (coming from the contribution of p-regular elements). Multiplying this by $|P|$ we get

$$|P| = \frac{\Lambda(1)^2}{|G:p|} + \sum{}' \Lambda_i(g)\overline{\Lambda_i(g)} + p^2|P|R. \tag{1}$$

Now, for $t = (|P| - 1)/s$,

$$\begin{aligned} \sum{}' \Lambda_i(g)\overline{\Lambda_i(g)} &= \sum{}' (\delta\lambda_i(g) + c)(\delta\lambda_i(g) + c) \\ &= \sum{}' \left(\lambda_i(g)\overline{\lambda_i(g)} + c\delta\left(\lambda_i(g) + \overline{\lambda_i(g)}\right) + c^2\right) \\ &= |P| - s - 2c\delta + c^2t. \end{aligned} \tag{2}$$

Since $|G : P| \equiv s \pmod{|P|}$, $\Lambda(1) \equiv \delta s + c \pmod{|P|}$ and $p|\Lambda(1)$,

$$\begin{aligned} \frac{\Lambda(1)^2}{|G:P|} \equiv \frac{(\delta s + c)^2}{s} \pmod{p\,|P|} &\equiv \frac{s^2 + 2\delta sc + c^2}{s} \\ &\equiv s + 2\delta c + \frac{c^2}{s}. \end{aligned} \tag{3}$$

Combining (1), (2) and (3) we find

$$0 \equiv c^2(1 + st) \equiv c^2|P| \pmod{p\,|P|}.$$

This forces $p|c$. Since $p|\Lambda(1)$ we also have $p|(\delta s + c)$, whence $p|s$, a contradiction. This proves the Theorem.

Finally, we remark that the Theorem is probably true more generally. For instance, the situation where $P \in \mathrm{Syl}_p(G)$ and $C_G(x) = C_G(P)$ for all $x \in P^{\#}$ could be tried. A proof similar to the above might work with the addition of some block calculations, as in [1, Part II].

References

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