A NOTE ON DIFFERENTIABLE FUNCTIONS

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Abstract. This paper provides a proof that the class of those real functions $f$ for which there exists a change of variable $g$ so that $f \circ g$ is differentiable coincides with the class of continuous functions which are of generalized bounded variation in the restricted sense.

The purpose of this note is to furnish the proof of a theorem stated in [1] which had been communicated privately to its authors. The definitions of $VBG_\ast$, $ACG_\ast$, Lusin’s condition (N) and Banach’s condition $(T_1)$ may be found in [2].

Theorem. A function $F$ can be transformed by means of an inner homeomorphism into an everywhere differentiable function if and only if $F$ is continuous and of generalized bounded variation in the restricted sense ($VBG_\ast$).

Proof. As noted in [1], the necessity of continuity and $VBG_\ast$ is obvious since these conditions are preserved under inner homeomorphisms and are satisfied by every differentiable function [2, p. 234]. Let $F$ be a continuous, $VBG_\ast$ function. Then $F$ satisfies Banach’s condition $(T_1)$ [2, p. 279]. Consequently, $F = G \circ H$, where $G$ is absolutely continuous and $H$ is of bounded variation [2, p. 287]. Bruckner and Goffman [1] have shown that there exists a homeomorphism $U$ such that $H \circ U$ is differentiable. Since differentiable functions satisfy condition (N) [2, Theorem 6.5, p. 227] and since $G$ is absolutely continuous, $F \circ U = G \circ (H \circ U)$ satisfies condition (N). As noted in the proof of necessity, $F \circ U$ is continuous and $VBG_\ast$. Thus $F \circ U$ is $ACG_\ast$ [2, Theorem 6.7, p. 227 and Theorem 8.8, p. 233]. Finally, Tolstov [3] has shown that each $ACG_\ast$ function can be transformed into a differentiable function by an inner homeomorphism.

REFERENCES

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