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A GENERAL MEASURE EXTENSION PROCEDURE

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ABSTRACT. We give a very general and flexible way of producing measure extensions. We obtain as corollaries many well-known and important measure extension and integral representation theorems as well as the main theorems of several recent papers.

Introduction. The question of when one can extend a measure (finitely, countably, or arbitrarily additive) is a very old one. Many different procedures over the years have been applied to obtain different types of significant extension theorems, mostly in topological spaces. Many of these have found numerous important applications in diverse branches of mathematics. Some of the more recent extension theorems have found applications in solving *topological* questions and are giving a great deal of insight into the interplay between measure and topology (see, in particular, [4], [5], [6]). In this paper we give a single, unifying procedure, which allows one to get simultaneously many of the well-known measure extension theorems. The method presented here is particularly useful for a variety of reasons: (1) It lends itself immediately to applications in the finite or totally finite cases. (2) It gives a great deal of flexibility in the actual construction of the measure extension. (3) In most of the important cases where the extension of the measure is not unique, this procedure gives *all* possible ways of extending the given measure. (4) The method ties in very naturally with many well-known representation theorems, and one obtains many of these as corollaries. (5) The actual construction of the measure extension is simple.

In the very simplest cases we obtain many nontrivial extension theorems for locally compact T_2 spaces. When our results are applied appropriately in many of the other cases, we obtain the main theorems of [4], [8], [9], [10], [12], as well as the important Alexandroff representation theorem [1]. Often the results obtained strengthen those in the literature and allow one to see clearly those situations in which the results extend to arbitrary topological spaces.

DEFINITIONS. By a paving $(X, \mathfrak{L})$ we mean a set X together with a lattice $\mathfrak{L}$ of subsets of X. We will suppress X and just say that $\mathfrak{L}$ is a paving. If $\emptyset \in \mathfrak{L}$ then $\mathfrak{L}$ is called a $\emptyset$ paving; if $X \in \mathfrak{L}$ also then $\mathfrak{L}$ is called $\emptyset$-X paving. If $\mathfrak{L}$ should be closed under countable intersections we say that $\mathfrak{L}$ is a delta paving.

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For T_2 topological spaces we use the following abbreviations:

(1) $\mathcal{K}_0$ is the paving of compact G_δ sets;

(2) $\mathcal{K}$ is the paving of compact sets;

(3) $\mathcal{Z}$ is the paving of zero sets (i.e. sets of the form $f^{-1}\{0\}$ where f is continuous);

(4) $\mathcal{F}$ is the paving of closed sets.

$\mathcal{L}$ is called compact if whenever $X \subset \bigcup L'_\alpha$, where $L_\alpha \in \mathcal{L}$, a finite subcover exists (L'_α denotes the complement of L_α relative to X). The definition of $\mathcal{L}$ is countably compact, $\mathcal{L}$ is normal, etc., are the same as in topological spaces, only we replace closed sets by sets in $\mathcal{L}$, and open sets by sets in $\{X - L\colon L \in \mathcal{L}\}$. (To achieve consistency with our definitions we follow those given in [3].) As an example, we have that in a T_2 space any subpaving of $\mathcal{K}$ is a compact paving. In any Tychonoff space the paving $\mathcal{Z}$ is a normal paving and is countably compact if and only if X is pseudocompact. This follows from the well-known Alexandroff-Glicksberg Theorem (see [17, p. 170]). If μ is a finitely additive measure on a ring R and $\mathcal{L} \subset R$ is some paving, μ is called $\mathcal{L}$ regular if $\mu(E) = \sup \mu(L)$, $L \subset E$ and $L \in \mathcal{L}$. If $R(\mathcal{L})$ represents the ring generated by the paving $\mathcal{L}$, we will denote by $MR(\mathcal{L})$ the collection of all finitely additive $\mathcal{L}$-regular measures defined on $R(\mathcal{L})$. $A(\mathcal{L})$ will denote the algebra generated by $\mathcal{L}$. If $\mu \in MR(\mathcal{L})$, μ is called σ-smooth if for any sequence $L_n \downarrow \varnothing$, where $L_n \in \mathcal{L}$, $\mu(L_n) \downarrow 0$. If $\mathcal{L}$ is a delta paving we will agree that σ-smooth measures are already defined on $S(\mathcal{L})$, the sigma ring generated by $\mathcal{L}$, since in this case we may always extend to $S(\mathcal{L})$ preserving $\mathcal{L}$-regularity. We will only work with nonnegative measures since this represents no loss of generality. $MR(\sigma, \mathcal{L})$ will denote the collection of σ-smooth measures in $MR(\mathcal{L})$. Of course, if μ is finite, $\mu \in MR(\sigma, \mathcal{L})$ if and only if μ is $\mathcal{L}$-regular and countably additive.

If $\mathcal{L}_1 \subset \mathcal{L}_2$ are lattices of subsets of X, we adopt the following definitions:

(a) $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$ if whenever $A \in \mathcal{L}_1$, $B \in \mathcal{L}_2$ and $A \cap B = \varnothing$, there exists a $C \in \mathcal{L}_1$ such that $B \subset C$ and $A \cap C = \varnothing$.

(b) $\mathcal{L}_1$ separates $\mathcal{L}_2$ if whenever $A \cap B = \varnothing$, where $A, B \in \mathcal{L}_2$, there exist $C, D \in \mathcal{L}_1$ such that $A \subset C$, $B \subset D$ and $C \cap D = \varnothing$.

(c) $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact ($\mathcal{L}_2$ is $\mathcal{L}_1$ c.b.) if whenever $B_n \downarrow \varnothing$, where $B_n \in \mathcal{L}_2$, $n = 1, 2, \ldots$, there exist $A_n \in \mathcal{L}_1$, $n = 1, 2, \ldots$, such that $B_n \subset A'_n \downarrow \varnothing$ ($B_n \subset A_n \downarrow \varnothing$).

If in the definition of $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact we let $\mathcal{L}_1 = \mathcal{L}_2$, we get the definition of $\mathcal{L}_1$ is countably paracompact. We will sometimes find it useful to apply definition (c) without the assumption that $\mathcal{L}_1 \subset \mathcal{L}_2$.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are pavings, we denote by $\mathcal{L}_1 \wedge \mathcal{L}_2$ the collection $\{L_1 \cap L_2\colon L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\}$. A linear functional Φ defined on a real vector space F of real valued functions is called nonnegative if whenever $f \geqslant 0$, $\Phi(f) \geqslant 0$.

If $\mathcal{L}$ is a $\varnothing$-X delta paving, $C_b(\mathcal{L})$ will denote the collection of all bounded $f\colon X \to R$ (the real line) such that $f^{-1}(C) \in \mathcal{L}$ for each closed subset C of R. $C_b(\mathcal{L})$ is a Banach algebra with identity under the sup norm, and it is easy to see that $C_b(\mathfrak{L}) = C_b(\mathfrak{Z}(\mathfrak{L}))$ where $\mathfrak{Z}(\mathfrak{L})$ is the collection of zero sets of functions in $C_b(\mathfrak{L})$. The following lemmas will be used throughout. The first is proved by a technique similar to the proof of the ordinary Urysohn Lemma and may be found in [1]; the other two are extremely easy to prove and may be found in [4], [5] or [15].

1.1 LEMMA. *If $\mathfrak{L}$ is a $\varnothing$-X delta paving which is normal, then whenever $A \cap B = \varnothing$, where A, $B \in \mathfrak{L}$, there is an $f \in C_b(\mathfrak{L})$ such that $f(A) = 1$, $f(B) = 0$, where $0 \leqslant f \leqslant 1$.*

1.2 LEMMA. *If $\mathfrak{L}_1 \subset \mathfrak{L}_2$ and $\mathfrak{L}_1$ semiseparates $\mathfrak{L}_2$, then if $\mu \in MR(\mathfrak{L}_2)$, its restriction to $R(\mathfrak{L}_1)$ is $\mathfrak{L}_1$-regular.*

1.3 LEMMA. *If $\mathfrak{L}_1 \subset \mathfrak{L}_2$ and $\mathfrak{L}_1$ separates $\mathfrak{L}_2$, then if $\mu \in MR(\mathfrak{L}_1)$ extends to a $\nu \in MR(\mathfrak{L}_2)$ finite on $\mathfrak{L}_2$, the extension is unique.*

2. **Assumptions.** We will assume throughout that $\mathfrak{L}_1 \subset \mathfrak{L}_2$ are both $\varnothing$ pavings, and that we are given $\mu \in MR(\mathfrak{L}_1)$ to begin with. B_1 will denote a vector space of $R(\mathfrak{L}_1)$ (measurable w.r.t. $R(\mathfrak{L}_1)$) integrable functions ($S(\mathfrak{L}_1)$ integrable functions if μ is σ-smooth), and $B_2 \supset B_1$ will denote an algebra of real valued $R(\mathfrak{L}_2)$ measurable functions. We form the linear functional Φ defined on B_1 as follows: $\Phi(f) = \int f\, d\mu$ where $f \in B_1$ and make the following assumptions throughout:

A(0) $\mathcal{C}$ is a $\varnothing$ paving $\subset \mathfrak{L}_2$.

A(1) Φ extends to a nonnegative linear functional Φ^* defined on B_2.

A(2) If $f \in B_2$, then $f^{-1}(-\infty, \alpha] \in \mathfrak{L}_2$ for $0 < \alpha < 1$.

A(3) For each $A \in \mathcal{C}$ there is an $f \in B_2$ such that $f \geqslant K_A$ (the characteristic function of A).

A(4) Whenever A, $B \in \mathcal{C}$ and $A \cap B = \varnothing$, there exists an $f \in B_2$ such that $f(A) = 1, f(B) = 0$ and $0 \leqslant f \leqslant 1$.

A(5) $\mathcal{C} \wedge \mathfrak{L}_2 \subset \mathcal{C}$.

Occasionally in applications the following condition holds:

(*) If $A \in \mathfrak{L}_1$, $B \in \mathcal{C}$ and $A \cap B = \varnothing$, there is an $f \in B_1$, such that $f(A) = 1, f(B) = 0$ and $0 \leqslant f \leqslant 1$.

3. **Construction.**

3.1 THEOREM. *Assuming* A(0)–A(5) *define ν on $\mathcal{C}$ as follows: For $A \in \mathcal{C}$, $\nu(A) = \inf \Phi^*(f)$, where $f \in B_2$ and $f \geqslant K_A$. Extend ν to $A(\mathfrak{L}_2)$ as follows: $\nu(E) = \sup \nu(A)$, where $A \subset E$ and $A \in \mathcal{C}$. Then ν is a measure.*

PROOF. The proof will be broken up for ease of reading. We first note that ν is clearly monotone; to show that ν is superadditive on 2^X it clearly suffices to show that ν is superadditive on $\mathcal{C}$.

(a) ν is superadditive on $\mathcal{C}$:

Suppose A, $B \in \mathcal{C}$ and $A \cap B = \varnothing$. Choose $\varepsilon > 0$ and $g \in B_2$ where $g \geqslant K_{A \cup B}$ and $\Phi^*(g) < \nu(A \cup B) + \varepsilon$. Choose by A(4) an $f \in B_2$ such that $f(A) = 1$ and $f(B) = 0$ where $0 \leqslant f \leqslant 1$. Let $g_1 = fg$ and $g_2 = g(1 - f)$.

Then $g_1 \geqslant K_A$, $g_2 \geqslant K_B$ and $g_1 + g_2 = g$. Thus

$$\nu(A) + \nu(B) \leqslant \Phi^*(g_1 + g_2) = \Phi^*(g) < \nu(A \cup B) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have that $\nu(A) + \nu(B) \leqslant \nu(A \cup B)$, and thus ν is superadditive on $\mathcal{C}$ and, hence, on 2^X.

(b) If E_1 and $E_2 \in 2^X$ and there exists an $F \in \mathcal{L}_2$ such that $F \supset E_1$ and $F \cap E_2 = \varnothing$, then $\nu(E_1 \cup E_2) = \nu(E_1) + \nu(E_2)$:

Choose $A \in \mathcal{C}$, $A \subset E_1 \cup E_2$. We note immediately that

$$A \cap F \subset E_1 \tag{1}$$

and, hence, that

$$A - F \subset E_2. \tag{2}$$

Using A(3) and A(5) choose $f \in B_2$, where $f \geqslant K_{A \cap F}$, and ε, where $0 < \varepsilon < 1$. Let $F_2 = f^{-1}(-\infty, 1 - \varepsilon]$. Then $F_2 \in \mathcal{L}_2$ and again we may choose a $g \in B_2$, such that $g \geqslant K_{A \cap F_2}$.

It is easy to see that on A, $f(x) + g(x) \geqslant 1 - \varepsilon$, and thus $(f(x) + g(x)) \cdot (1 - \varepsilon)^{-1} \geqslant K_A$. Thus $\Phi^*(f) + \Phi^*(g) \geqslant (1 - \varepsilon)\nu(A)$. Letting $\varepsilon \to 0$ and g vary we get that

$$\Phi^*(f) + \nu(A \cap F_2) \geqslant \nu(A). \tag{3}$$

By the definition of F_2 we have $F_2 \cap (F \cap A) = \varnothing$, and thus, $A \cap F_2 \subset F'$. It follows using (2) that $A \cap F_2 \subset A - F \subset E_2$. Thus, replacing $\nu(A \cap F_2)$ in (3) by the larger quantity $\nu(E_2)$ and letting f vary, we get

$$\nu(A \cap F) + \nu(E_2) \geqslant \nu(A)$$

and it follows from (1), the monotonicity of ν, and the arbitrariness of A that

$$\nu(E_1) + \nu(E_2) \geqslant \nu(E_1 \cup E_2).$$

Combining this with superadditivity of ν on 2^X we get the desired result.

Now let $\mathcal{H} = \{A \in 2^X\colon \nu(T) = \nu(T \cap A) + \nu(T - A) \text{ for all } T \in 2^X\}$. Then standard arguments show that ν is additive on $\mathcal{H}$, and that $\mathcal{H}$ is an algebra.

(c) $A(\mathcal{L}_2) \subset \mathcal{H}$:

We need only show that $\mathcal{L}_2 \subset \mathcal{H}$. However, this is simple, for if $F \in \mathcal{L}_2$ and T is arbitrary, $T \cap F \subset F$ and $(T - F) \cap F = \varnothing$, and, hence, by part (b),

$$\nu(T) = \nu((T - F) \cup (T \cap F)) = \nu(T - F) + \nu(T \cap F). \quad \square$$

3.2 *Note.* If $\mathcal{C} \subset \mathcal{L}_1$, and if we can show that ν extends μ, then ν will be $\mathcal{L}_1$-regular.

3.3 *Note.* If $\mathcal{C}$ is $\mathcal{L}_1$ countably paracompact, or $\mathcal{L}_1$ c.b., or compact, or countably compact, then if ν extends μ and μ is σ-smooth, ν is σ-smooth. To see this suppose $B_n \downarrow \varnothing$ where $B_n \in \mathcal{L}_2$. Then there exist $A_n \in \mathcal{C}$, where $A_n \subset B_n$ and $\nu(B_n) - \nu(A_n) < \varepsilon/2^n$. We can, of course, arrange things so that $A_n \downarrow \varnothing$. By $\mathcal{L}_1$ countable paracompactness for example, there exist $D_n \in \mathcal{L}_1$ such that $A_n \subset D_n' \downarrow \varnothing$. Since μ is $\mathcal{L}_1$ regular, $\mu(D_n') \downarrow 0$, hence $\nu(D_n') \downarrow 0$ and, hence, $\nu(A_n)\downarrow 0$. It follows that $\nu(B_n)\downarrow 0$. The proof of the remaining parts are simple.

3.4 *Note*. Nowhere in the actual *construction* of ν did we use the $\mathfrak{L}_1$ regularity of μ, or the measurability of the functions in B_2.

4. **Some applications.** It remains to establish the relationship between ν and μ. We do this first in some special cases.

4.1 THEOREM. *If X is a locally compact T_2 space and if $B_1 = B_2 =$ the collection of continuous functions of compact support, $\Phi = \Phi^*$, $\mathfrak{L}_1 = \mathcal{K}_0$, and $\mathfrak{L}_2 = \mathfrak{F}$, then ν extends μ. Thus in a locally compact T_2 space every $\mu \in MR(\mathcal{K}_0)$ extends to a $\nu \in MR(\sigma, \mathfrak{F})$. The extension is even $\mathcal{K}$ regular.*

PROOF. We need only take $\mathcal{C} = \mathcal{K}$. A(3) and A(4) are immediate from [7, Theorem 3, p. 174]. All the other assumptions are trivial to verify. To see that ν is σ-smooth, we need only note that $\mathcal{C}$ is a compact lattice. To see that ν actually extends μ, we note that for $D \in \mathcal{K}_0$,

$$\nu(D) = \inf \Phi^*(f) = \inf \Phi(f) = \inf \int f\, d\mu = \mu(D)$$

where $f \in B_1 = B_2$. The last equality follows from [7, Theorem 2, p. 175] and Lebesgue's dominated convergence theorem since $\int f\, d\mu$ is finite. Thus $\mu = \nu$ on $\mathfrak{L}_1$, hence on $R(\mathfrak{L}_1)$, and hence on $\sigma(\mathfrak{L}_1)$. □

COROLLARY. *If X is a locally compact T_2 space then every $\mu \in MR(\mathcal{K}_0)$ $(= MR(\sigma, \mathcal{K}_0))$ extends to a $\nu \in MR(\sigma, \mathcal{K})$.*

4.2 THEOREM. *If X is a locally compact T_2 space, then every $\mu \in MR(\sigma, \mathcal{K}_0)$ extends to a $\nu \in MR(\sigma, \mathfrak{Z})$. The extension is even $\mathcal{K}_0$ regular.*

PROOF. Again take $\Phi^* = \Phi$, $B_1 = B_2 =$ the collection of continuous functions of compact support, $\mathfrak{L}_1 = \mathcal{C} = \mathcal{K}_0$, and $\mathfrak{L}_2 = \mathfrak{Z}$. Again A(0)–A(4) are trivial and A(5) is simple since $\mathcal{K}_0 \cap \mathfrak{Z} = \mathcal{K}_0$. We prove ν extends μ in the same way as in the previous theorem. Actually, this extension theorem is a corollary of 4.1 but not the assertion of $\mathcal{K}_0$ regularity. □

COROLLARY. *If X is a locally compact T_2 space, then every $\mu \in MR(\sigma, \mathcal{K})$ extends to a $\nu \in MR(\sigma, \mathfrak{F})$.*

Restrict μ to $\sigma(\mathcal{K}_0)$. By Lemma 1.2 and the Baire Sandwich Theorem (see [7, p. 176]), the restriction $\mu_1 \in MR(\mathcal{K}_0)$. Now use 4.1 to extend μ to $\nu \in MR(\sigma, \mathfrak{F})$. $\nu|_{\sigma(\mathcal{K})}$ must $= \mu$ since, by Lemma 1.3, the extension of μ to $\sigma(\mathcal{K})$ is unique.

Alternatively one could have obtained this by taking $\mathfrak{L}_1 = \mathcal{C} = \mathcal{K}$, $\mathfrak{L}_2 = \mathfrak{F}$ and using an argument similar to 4.2. Another simple proof of this occurs in [8]. □

The following will be useful for later applications:

4.4 LEMMA. *Assuming* A(0)–A(5), $\mathcal{C} \supset \mathfrak{L}_1$, *and condition* (*), *then if μ is finite on $A(\mathfrak{L}_1)$, and $\Phi^*(f^*) = \int f^*\, d\nu$ for every $f^* \in B_2$, then ν extends μ.*

Proof. Since μ is $\mathcal{L}_1$ regular we may choose for any $A \in \mathcal{L}_1$, and $\varepsilon > 0$ a $B \in \mathcal{L}_1$ such that $B \subset A'$ and $\mu(A' - B) < \varepsilon$. Since $\mathcal{C} \supset \mathcal{L}_1$ we may choose by (*) an $f \in B_1$ such that $f(A) = 1$ and $f(B) = 0$ where $0 \leqslant f \leqslant 1$. Since $f \in B_2$ we have

$$\nu(A) \leqslant \int f\, d\nu = \Phi^*(f) = \Phi(f) = \int f\, d\mu$$

$$= \int_B f\, d\mu + \int_{A'-B} f\, d\mu + \int_A f\, d\mu \leqslant \mu(A) + \varepsilon.$$

Thus $\nu \leqslant \mu$ on $\mathcal{L}_1$. The reverse inequality is proved in an identical manner interchanging the roles of ν and μ, Φ^* and Φ and using the $\mathcal{C}$ regularity of ν. Thus $\nu = \mu$ on $\mathcal{L}_1$, hence on $A(\mathcal{L}_1)$. □

The condition $\Phi^*(f^*) = \int f^*\, d\nu$ is important. We give some conditions under which this holds.

4.5 **Lemma.** *If B_2 is a subalgebra of $C_b(\mathcal{L}_2)$ containing constants, and if* A(0)–A(5) *hold with $\mathcal{C} = \mathcal{L}_2 \cup \{X\}$, then $\Phi^*(f^*) = \int f^*\, d\nu$.*

Proof. Suppose first that $f^* \geqslant 0$. By linearity we may suppose that $\sup f^*(x) < 1$.

If $F_k = (f^*)^{-1}[(k/n), \infty)$, where $k = 0, 1, 2, \ldots, n$, then $F_k \in \mathcal{L}_2$, and by a standard argument

$$\int f^*\, d\nu \geqslant n^{-1} \sum_1^n \nu(F_k). \tag{1}$$

We may choose for each $k = 1, 2, \ldots, n$, $f_k^* \geqslant K_{F_k}$, where $f_k^* \in B_2$, $\nu(F_k) > \Phi^*(f_k^*) - n^{-1}$. We have, therefore,

$$n^{-1} \sum_1^n \nu(F_k) \geqslant \Phi^*\left(n^{-1} \sum_1^n f_k^* \right) - n^{-1}. \tag{2}$$

Since $\sum_1^n f_i^*(x) \geqslant k$ on F_k, it follows that on $F_k - F_{k-1}$, $k = 1, 2, \ldots, n$,

$$n^{-1}\left(\sum_1^n f_i^*(x) \right) + n^{-1} \geqslant (k+1)/n > f^*(x),$$

and, hence, by the nonnegativity of Φ^*, that

$$\Phi^*\left(n^{-1} \sum_1^n f_i^*(x) \right) + Nn^{-1} \geqslant \Phi^*(f^*) \tag{3}$$

where $N = \Phi^*(1)$. Combining (3) with (2) and using (1) we get that $\int f^*\, d\nu \geqslant \Phi^*(f^*) - (N+1)n^{-1}$. Since n was arbitrary, $\int f^*\, d\nu \geqslant \Phi^*(f^*)$.

To prove the inequality the other way, we note that $\nu(X) = \Phi^*(1)$. If we call $\int f^*\, d\nu = \Psi^*(f^*)$, then it follows from the preceding that $\Phi^*(f^* - \inf f^*) \leqslant \Psi^*(f^* - \inf f^*)$, where $f^* \in B_2$ is arbitrary. And noting that $\Phi^*(1) = \Psi^*(1)$, we have, by linearity, $\Phi^*(f^*) \leqslant \Psi^*(f^*)$ for any $f^* \in B_2$. Replacing f^* by its negative we get the inequality the other way. □

5. **Further applications.** We may now give some other applications. The first theorem is the main theorem of [4]. It has as corollaries both the main theorems of [9] and [12], and has found numerous measure theoretic and topological corollaries (see [5], [6] for details). We assume that all measures in this section are bounded.

5.1 THEOREM. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are $\emptyset$-X delta normal pavings, and if $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$, then if $\mu \in MR(\mathcal{L}_1)$, μ extends to a $\nu \in MR(\mathcal{L}_2)$. If μ is σ-smooth and $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact, or $\mathcal{L}_1$ c.b., or compact, or countably compact, then if μ is σ-smooth so is ν. If $\mathcal{L}_1$ separates $\mathcal{L}_2$, then every $\mu \in MR(\mathcal{L}_1)$ extends uniquely to a $\nu \in MR(\mathcal{L}_2)$ and conversely.*

PROOF. Take $\mathcal{C} = \mathcal{L}_2$, $B_1 = C_b(\mathcal{L}_1)$ and $B_2 = C_b(\mathcal{L}_2)$. Then since B_2 is a locally convex T_2 topological vector space with the norm topology, we may extend Φ by Krein's theorem [13, p. 63] to a nonnegative Φ^* defined on $C_b(\mathcal{L}_2)$. Thus A(1) is verified.

All the other axioms are easily verified and, furthermore, condition (∗) holds since $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$ and $\mathcal{L}_1$ is normal. Using Lemma 4.5 we have that $\Phi^*(f^*) = \int f^* \, d\nu$ and the result follows from Lemma 4.4. □

One notes that the use of B_1 in the construction was really auxiliary and used only to construct a linear functional. Thus, if we suppress knowledge of μ and begin with a linear functional on B_1, we may construct ν as before, and under mild conditions $\Phi^*(f^*) = \int f^* \, d\nu$. If we begin therefore with a $\emptyset$-X delta normal paving and a nonnegative linear functional Φ on $B_1 = C_b(\mathcal{L}_1)$, then by taking $B_1 = B_2$, $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{C} = \mathcal{L}$, $\Phi^* = \Phi$, we have, using Lemma 4.5, the following main theorem of Alexandroff [2].

5.2 THEOREM. *If $\mathcal{L}$ is a $\emptyset$-X delta normal paving and Φ is a nonnegative linear functional on $C_b(\mathcal{L})$, then $\Phi(f) = \int f \, d\nu$ for some (unique) $\nu \in MR(\mathcal{L})$.*

Actually Alexandroff proves this theorem for bounded linear functionals, but clearly any such functional can be written as the difference of two nonnegative linear functionals. The uniqueness in the above theorem is easily proven by an argument similar to the proof of Lemma 4.4. It appears that 5.2 can be generalized, since we do not have to take B_1 = all of $C_b(\mathcal{L}_1)$. Any subalgebra of $C_b(\mathcal{L}_1)$ which satisfies A(4) will do (all the other axioms are trivial to verify). However the generality in this case is only apparent. For if B_1 is not uniformly closed one can always pass to the uniform closure, and under very mild separation conditions the uniform closure is all of $C_b(\mathcal{L}_1)$ (see [14]).

As another corollary in this direction, we have the following main theorem of Kirk [10].

5.3 THEOREM. *Let A be a uniformly closed algebra of bounded real valued functions on X which contains constants and separates points of X. Let $\mathcal{U}(A)$ be any $\emptyset$-X paving which is a base for a weak topology generated by A and which contains the zero sets of A. Then if, whenever $C, D \in \mathcal{U}(A)$, there exists* *an $f \in A$ such that $f(C) = 1$ and $f(D) = 0$, then the dual of A is isomorphic as a Banach lattice to $MR(\mathfrak{A}(A))$, the isomorphism is given by $\Phi \to m$ where $\Phi(f) = \int f\, dm$.*

PROOF. Take $B_1 = B_2 = A$, $\mathfrak{L}_1$ = zero sets of A, $\mathfrak{L}_2 = \mathfrak{A}(A) = \mathcal{C}$, $\Phi^* = \Phi$.

A(2) follows from the well-known fact that A must be lattice and from the fact that $f^{-1}(-\infty, \alpha] = Z((f - \alpha) \wedge 0) \in \mathfrak{L}_1 \subset \mathfrak{L}_2$, where $Z(f)$ represents the zero set of f. All other axioms are trivial to verify. □

The proof of 5.3 given in [10] is very long. 5.3, as well as the modification of 5.3 given in [11], can both be proved in a much shorter manner by going to the Wallman compactification and using thickness type arguments. (See [15] for details.)

The following final application (see [16] in this connection) gives a very general theorem and shows in a sense how far some of the previous theorems can be generalized in finite situations.

5.4 THEOREM. *Suppose $\mathfrak{L}$ is a $\varnothing$-X delta paving and $\mu \in MR(\sigma, \mathfrak{L})$ (is finite); then μ extends to a $\nu \in MR(\sigma, \mathfrak{F})$ where $\mathfrak{F} = \{F \subset X | F \cap L \in \mathfrak{L}$ for all $L \in \mathfrak{L}\}$. The extension is even $\mathfrak{L}$-regular.*

PROOF. We need only take $\mathfrak{L}_1 = \sigma(\mathfrak{L}) = \mathcal{C}$, $\mathfrak{L}_2 = \sigma(\mathfrak{F})$, $B_1 = B_2 =$ bounded μ integrable functions, $\Phi^* = \Phi$. A(0)–A(5) are trivial; thus $\nu \in MR(\mathfrak{L}_2)$.

To see ν extends μ note that for $A \in \mathfrak{L}_1$

$$\nu(A) = \inf \Phi^*(f) = \inf \Phi(f) = \int K_A\, d\mu = \mu(A);$$

hence $\nu = \mu$ on $\sigma(\mathfrak{L}_1)$.

That ν is σ-smooth follows from Note 3.3. $\mathfrak{L}$-regularity follows from Note 3.2. □

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