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A UNIVALENT FUNCTION NOWHERE SEMICONFORMAL ON THE UNIT CIRCLE

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ABSTRACT. We shall construct a function f holomorphic and univalent in the open unit disk such that f is not semiconformal at any point of the unit circle. It is also shown that f may be extended quasiconformally to the whole extended plane.

1. Introduction. Let S be the family of all functions holomorphic and univalent in $D = \{|z| < 1\}$. Then $f \in S$ is called conformal at a point ζ of $\Gamma = \{|z| = 1\}$ if f has the angular limit $f(\zeta) \neq \infty$ at ζ and if the function $\arg[(f(\zeta) - f(z))/(\zeta - z)]$ of z has a finite angular limit at ζ (see [**5**, p. 303]). Following D. M. Campbell and J. A. Pfaltzgraff [**2**, p. 74], we call $f \in S$ semiconformal at $\zeta \in \Gamma$ if the radial limit $f^*(\zeta) = \lim_{r\to 1-} f(r\zeta) \neq \infty$ exists, and if the function $(f^*(\zeta) - f(z))/((\zeta - z)f'(z))$ of z has the angular limit one at ζ. It is known that if f is conformal at ζ, then f is semiconformal at ζ [**6**], while the converse is false ([**7**, p. 85 with a few modifications]; see [**4**, p. 258] also). It is also known that *there exists $f \in S$ such that f is not conformal at any point of* Γ [**5**, p. 304]. We shall prove the following theorem which extends the above italicized proposition.

THEOREM. *There exists f holomorphic and univalent in D such that f is not semiconformal at any point of* Γ.

2. Proof of Theorem.

LEMMA 1. *A function $f \in S$ is semiconformal at $\zeta \in \Gamma$ if and only if*

$$\lim(\zeta - z)f''(z)/f'(z) = 0 \tag{2.1}$$

as $z \to \zeta$ in each Stolz angle at ζ.

To prove Lemma 1 it suffices to consider the case $\zeta = 1$. We note that f is semiconformal at 1 if and only if the same is true of F defined by

$$F(z) = (f(z) - f(0))/f'(0), \qquad z \in D.$$

According to C. Pommerenke [**4**, Theorem 3.15, p. 257] F is semiconformal at 1 if and only if

$$F(z, \xi) \to z/(1 + z) \tag{2.2}$$

as $0 < \xi \to 1 -$ locally uniformly in D, where

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$$F(z, \xi) = \frac{F((z + \xi)/(1 + \xi z)) - F(\xi)}{(1 - \xi^2)F'(\xi)};$$

in effect, Pommerenke [**4**, p. 257] adopted (2.2) as the definition of semiconformality at 1. Lemma 1 now follows from the first half of [**4**, Theorem 3.14, p. 255], combined with the identity $F''/F' = f''/f'$.

LEMMA 2 (P. LAPPAN [3, p. 113]). *There exists a function g holomorphic in D which satisfies the following*:

$$\sup_{z \in D} (1 - |z|)|g(z)| \leqslant 2; \tag{2.3}$$

$$\limsup_{r \to 1-} (1 - r)|g(r\zeta)| > 0 \tag{2.4}$$

at each $\zeta \in \Gamma$.

For the proof of our theorem we set

$$f(z) = \int_0^z \exp\left(\int_0^w g(\eta)/5 d\eta\right) dw$$

so that

$$f''/f' = g/5. \tag{2.5}$$

Then it follows from (2.3) and (2.5) that

$$\sup_{z \in D} (1 - |z|^2)|f''(z)/f'(z)| \leqslant 4/5 < 1. \tag{2.6}$$

Therefore f is univalent in D by the result of J. Becker [**1**, Corollary 4.1, p. 36]. Furthermore, (2.1) is false at each $\zeta \in \Gamma$ because of (2.4) and (2.5). We now conclude that f is not semiconformal at any point of Γ.

REMARK. Actually, f with (2.6) admits a quasiconformal extension to the whole extended plane by the cited result of Becker [**1**, Corollary 4.1, p. 36]. Therefore *quasiconformality does not imply semiconformality*.

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