PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 1, April 1978

QUASITRIANGULAR MATRICES

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ABSTRACT. It is shown that there exist quasitriangular operators which cannot be represented as quasitriangular matrices.

Introduction. A quasitriangular matrix is an infinite matrix $A = [a_{ij}]$ in which all entries below the subdiagonal are zero and the subdiagonal entries cluster at zero (i.e., $a_{ij} = 0$ for $i > j + 1$ and $\lim\inf|a_{j+1,j}| = 0$). A quasitriangular operator is an operator which can be expressed as the sum of a triangular matrix ($a_{ij} = 0$ for $i > j$) and a compact one. The relationship between quasitriangular matrices and quasitriangular operators, and the significance of studying that relationship in conjunction with the invariant subspace problem, are discussed by Halmos in [2]. It is shown in [2] that every bounded quasitriangular matrix defines a quasitriangular operator. Halmos then asks whether every cyclic quasitriangular operator has a quasitriangular matrix. It will be shown below that the answer is no. A few preliminary ideas are needed.

Let $A = \int \eta \, dE_\eta$ be a bounded selfadjoint operator defined on a separable Hilbert space $\mathcal{H}$. By Weyl's theorem A is the sum of a diagonal operator and a compact one. Hence A is quasitriangular. Denote by $\mathcal{H}_a(A)$ the set of elements x in $\mathcal{H}$ for which $\|E_\eta x\|^2$ is an absolutely continuous function of η. The subspace $\mathcal{H}_a(A)$ reduces A [1, p. 104] and the restriction of A to $\mathcal{H}_a(A)$ is called the absolutely continuous part of A. A result due to Kato [3] and Rosenblum [4] asserts that the absolutely continuous part of the operator A remains stable under a trace class perturbation. In particular, if C is selfadjoint and of trace class, and if $B = A + C$, then the absolutely continuous parts of A and B are unitarily equivalent.

Main result. The main result to be established is as follows.

PROPOSITION. *A selfadjoint operator with a nontrivial absolutely continuous part cannot be represented as a quasitriangular matrix.*

PROOF. Let A be a selfadjoint operator with a nontrivial absolutely continuous part. Suppose that with respect to some orthonormal basis, A can be represented as a quasitriangular matrix. Clearly this matrix takes the form

Received by the editors August 22, 1977.
AMS (MOS) subject classifications (1970). Primary 47B15.

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$$\begin{bmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ 0 & 0 & a_3 & b_4 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

with some subsequence of $\{a_n\}$ converging to zero. Furthermore, the subsequence $\{a_{n_k}\}$ can be chosen so that $\Sigma|a_{n_k}| < \infty$.

Let B be the matrix obtained from A by replacing each a_{n_k} by zero. Then B has finite dimensional invariant subspaces. In fact, B has a pure point spectrum.

If $C = A - B$ then C is the real part of a weighted shift with weight sequence $\{c_n\}$ satisfying $\Sigma|c_n| < \infty$. Hence C is of trace class. Since A has an absolutely continuous part it follows, from the Kato-Rosenblum theorem, that $A - C = B$ has an absolutely continuous part. But this contradicts the fact that B has a pure point spectrum.

COROLLARY. *The real part of the unilateral shift does not have a quasitriangular matrix.*

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