CR SUBMANIFOLDS OF A KAHLER MANIFOLD. I

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ABSTRACT. The differential geometry of CR submanifolds of a Kaehler manifold is studied. Theorems about totally geodesic CR submanifolds and totally umbilical CR submanifolds are given.

1. Introduction. Many papers have been concerned with complex submanifolds of complex manifolds, especially of complex space forms (see [4] for a survey of results). Recently, B.Y. Chen and K. Ogiue [1] have studied totally real submanifolds of complex manifolds. Later these submanifolds were further investigated by K. Yano, M. Kon, G. D. Ludden and M. Okumura [3], [6], [7].

The purpose of this paper is to initiate a study of a new class of submanifolds of a complex manifold. In §2 we introduce the concept of CR submanifold and we give its basic properties. CR submanifolds have been studied, till now, only from the analytic viewpoint (i.e. concerning the complex structure). Different kinds of sectional curvature, Ricci tensor and scalar curvature of a CR submanifold of a complex space form are examined in §§3 and 4. Also, some results on totally geodesic CR submanifolds and totally umbilical CR submanifolds are proved.

2. CR submanifolds. Let $N$ be a Kaehler manifold of complex dimension $n$ and $M$ be an $m$-dimensional Riemannian submanifold immersed in $N$. Denote by $g$ (resp. $g_0$) the Kaehlerian metric on $N$ (resp. the Riemannian metric on $M$), by $J$ the almost complex structure on $N$ and by $\varphi$ the isometric immersion of $M$ into $N$.

In order to simplify the presentation, we identify, for each $x \in M$, the tangent space $T_xM$ with $\varphi^*(T_xM) \subset T_{\varphi(x)}N$ by means of $\varphi$. The Riemannian metric $g_0$ is identified with the restriction of $g$ to the subspace $\varphi^*(T_xM)$. With this identification in mind we drop the symbol $g_0$, using instead the symbol $g$.

Now, suppose on $M$ a differentiable distribution $D: x \to D_x \subset T_xM$ ($\dim D_x = 2p$) is given. This distribution is assumed to be consistent with the almost complex structure on $N$, that is, $J(D_x) = D_x$ for each $x \in M$. Moreover, the complementary orthogonal distribution $D^\perp: x \to D^\perp_x \subset T_xM$
(dim \( D_x^\perp = q \)) is supposed to be totally real, that is, \( J(D_x^\perp) \subset v_x \) for each \( x \in M \), where \( v_x \) is the normal space to \( M \) at \( x \).

The distribution \( D \) (resp. \( D^\perp \)) can be defined by a projector \( P \) (resp. \( Q \)) which satisfy the well-known conditions

\[
P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad g \circ (P \times Q) = 0.
\]

We call the distribution \( D \) (resp. \( D^\perp \)) the horizontal (resp. vertical) distribution on \( M \).

**Definition.** The submanifold \( M \) endowed with the above pair of distributions \((D, D^\perp)\) is called a CR submanifold of \( N \).

**Remarks.** 1. Any real curve or real hypersurface of \( N \) is automatically a CR submanifold.
2. If, in particular, \( \dim D_x^\perp = 0 \) (resp. \( \dim D_x = 0 \)) for any \( x \in M \), the CR submanifold \( M \) is a complex submanifold (resp. totally real submanifold) of \( N \).

If \( \xi \) is a vector field in the normal bundle, put

\[
J\xi = A\xi + B\xi + C\xi
\]

where \( A\xi \) (resp. \( B\xi \)) is the horizontal (resp. vertical) part of \( J\xi \) and \( C\xi \) the normal part. Thus, \( A \) (resp. \( B \)) is a horizontal (resp. vertical) valued 1-form on the normal bundle and \( C \) is an endomorphism of the normal bundle.

If \( X \) is a vector field on \( M \), then \( JQX \) is a section in the normal bundle of \( M \), and from (2.2) we have

\[
BJQX + QX = 0,
\]

\[
AJQX = CJQX = 0.
\]

Applying \( J \) to (2.2) and comparing horizontal, vertical and normal parts we obtain

\[
C^2\xi + JB\xi + \xi = 0,
\]

\[
JA\xi + AC\xi = 0,
\]

\[
BC\xi = 0.
\]

From (2.4) and (2.5) we get \( C^3 + C = 0 \) on the normal bundle, that is the structure introduced by K. Yano [5].

Let \( \nabla \) be the Kaehlerian connection on \( N \). The Gauss and Weingarten equations are

\[
\nabla_X Y = \nabla_X Y + h(X, Y),
\]

\[
\nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi,
\]

where \( \nabla \) is the Riemannian connection on \( M \), \( \nabla^\perp \) is the connection on the normal bundle induced by \( \nabla \) and \( h \) is the second fundamental form of the immersion. \( A_\xi \) is an endomorphism of the tangent bundle of \( M \), and cannot be confused with the 1-form \( A \) defined by (2.2).

Since \( \nabla \) is a Kaehlerian connection, we have

\[
P(\nabla_X JPY) - P(A_{JQY}X) = JP(\nabla_X Y) + Ah(X, Y),
\]
(2.11) \( Q(\nabla_X JPY) - Q(A_{JQY} X) = Bh(X, Y) \),

(2.12) \( h(X, JPY) + \nabla^J_X JQY = JQ(\nabla_X Y) + Ch(X, Y) \)

for all vector fields \( X, Y \) on \( M \).

Differentiating (2.2) and comparing horizontal, vertical and normal parts we obtain

(2.13) \( P(\nabla_X A\xi) + P(\nabla_B A\xi) + JP(A\xi X) = P(A_{C\xi} X) + A(\nabla_X \xi) \),

(2.14) \( Q(\nabla_X A\xi) + Q(\nabla_B A\xi) = Q(A_{C\xi} X) + B(\nabla_X \xi) \),

(2.15) \( h(X, A\xi) + h(X, B\xi) + \nabla^J_X C\xi + J(QA\xi X) = C(\nabla_X A\xi) \)

for each vector field \( X \) on \( M \) and normal section \( \xi \).

3. Sectional curvature of a \( CR \) submanifold. Suppose now that \( N \) is a complex space form of constant holomorphic curvature \( c \). Then, the curvature tensor \( R \) of \( N(c) \) is given by

\[
R(X, Y)Z = \frac{c}{4} \left\{ g(Y, Z)X - g(X, Z)Y + g(JPY, Z)JX - g(JPX, Z)JY + 2g(X, JPY)JZ \right\}.
\]

(3.1)

The equation of Gauss becomes

\[
g(R(X, Y)Z, W) = \frac{c}{4} \left\{ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(JPY, Z)g(JPX, W) - g(JPX, Z)g(JPY, W) + 2g(X, JPY)g(JPZ, W) \right\}
\]

\[+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)).\]

(3.2)

The sectional curvature \( K_M \) of \( M \) determined by orthonormal vectors \( X \) and \( Y \) is given by

\[
K_M(X \wedge Y) = \frac{c}{4} \left\{ 1 + 3g(PX, JPY)^2 \right\} + g(h(X, X), h(Y, Y))
\]

(3.3)

\[-g(h(X, Y), h(X, Y)).\]

Definition. The holomorphic sectional curvature \( H \) of \( M \) determined by a unit vector \( X \in D \) is the sectional curvature determined by \{\( A_X, JA_X \)\}. Hence from (3.3) we have

(3.4) \( H(X) = c + g(h(X, X), h(JX, JX)) - g(h(X, JX), h(X, JX)). \)

From (2.12) we have

(3.5) \( h(X, JY) = JQ(\nabla_X Y) + Ch(X, Y) \)

for any two vector fields on \( M \) which lie in \( D \) (i.e. \( X_x, Y_x \in D_x, \forall x \in M \)). As a consequence of (3.5) and (2.4) we obtain

(3.6) \( h(JX, JY) = JQ(\nabla_JX Y) + C^2 h(X, Y) \quad \forall X, Y \in D. \)
Therefore, the holomorphic sectional curvature $H$ of $M$ determined by a unit vector $X \in D$ is given by

\[
H(X) = c + \|Bh(X, X)\|^2 - \|h(X, X)\|^2 - \|Ch(X, X)\|^2
- \|Q \nabla_X X\|^2 + g(h(X, X), JQ \nabla_X X)
- 2g(Ch(X, X), JQ \nabla_X X).
\]

(3.7)

**Remark.** If, in particular, $M$ is a complex submanifold of $N$ (i.e. $Q = 0$, $J = C$), then we get the known formula for its holomorphic sectional curvature [4]:

\[
H(X) = c - 2\|h(X, X)\|^2.
\]

**Definition.** The horizontal distribution $D$ is called parallel with respect to the Riemannian connection $\nabla$ on $M$ if $\nabla_X Y \in D$ for any two vector fields $X$, $Y \in D$.

**Theorem 1.** If $M$ is a CR submanifold of a complex space form $N(c)$ and $D$ is an involutive distribution, then the holomorphic sectional curvature verifies $H(X) < c \forall X \in D$.

**Proof.** Since $\nabla$ is the Levi-Civita connection on $M$ and $D$ is involutive, from (3.5) we have

\[
h(X, JY) - h(JX, Y) = JQ(\nabla_X Y - \nabla_Y X) = JQ([X, Y]) = 0.
\]

Hence $h(JX, JY) = - h(X, Y)$, and the assertion follows from (3.4).

**Remarks.** 1. The distribution $D$ is involutive, if and only if, $h(X, JY) = h(JX, Y) \forall X, Y \in D$.

2. If $D$ is parallel with respect to $\nabla$, then it is involutive and Theorem 1 is also valid.

**Definition.** The CR submanifold $M$ is called $D$-totally geodesic (resp. $D^\perp$-totally geodesic) if $h(X, Y) = 0$ for each $X, Y \in D$ (resp. $X, Y \in D^\perp$).

Then from (3.4) we have

**Theorem 2.** A CR submanifold $M$ of a complex space form $N(c)$ is $D$-totally geodesic if and only if the following conditions are fulfilled:

1. The horizontal distribution is involutive.
2. $H(X) = c$ for each $X \in D$.

Now, let $\{E_1, \ldots, E_m\}$ be a local field of orthonormal frames on $M$ such that $\{E_1, \ldots, E_p, E_{p+1} = JE_1, \ldots, E_{2p} = JE_p\}$ (resp. $\{E_{2p+1}, \ldots, E_{2p+q}\}$) is a local field of frames in $D$ (resp. $D^\perp$).

**Definition.** The CR submanifold $M$ is called $D$-minimal (resp. $D^\perp$-minimal, if $\sum_{i=1}^{2p} h(E_i, E_i) = 0$) (resp. $\sum_{i=1}^{2p} h(E_{2p+i}, E_{2p+i}) = 0$).

**Remark.** Every CR submanifold with involutive horizontal distribution is $D$-minimal.

**Theorem 3.** If $M$ is a $D^\perp$-minimal CR submanifold of a complex space form $N(c)$, then $M$ is $D^\perp$-totally geodesic, if and only if,
\( (3.8) \quad K_M(X \wedge Y) = c/4 \quad \forall X, Y \in D^\perp. \)

**Proof.** Substitute \( X \) and \( Y \) from (3.3) by \( QX \) and \( QY \) and obtain
\[
K_M(X \wedge Y) = c/4 + g(h(QX, QX), h(QY, QY))
- g(h(QX, QY), h(QX, QY)).
\]

Supposing (3.8) valid and taking into account the \( D^\perp \)-minimality of \( M \), from (3.9) we have
\[
g(h(E_{2p+i}, E_{2p+j}), h(E_{2p+i}, E_{2p+j})) = 0 \quad \forall 1 < i, j < q.
\]
Therefore \( h(X, Y) = 0 \) for each vector field \( X, Y \) which lies in \( D^\perp \). Of course, if \( M \) is \( D^\perp \)-totally geodesic, (3.8) follows from (3.9).

**Definition.** A CR submanifold \( M \) is called CR totally geodesic, if \( h(X, Y) = 0 \) for any \( X \in D \) and \( Y \in D^\perp \). The sectional curvature determined by orthonormal vectors \( X \in D \) and \( Y \in D^\perp \) is called CR sectional curvature.

**Theorem 4.** If the CR sectional curvature of \( M \) is given by
\[
K_M(X \wedge Y) = c/4 \quad \forall X \in D, Y \in D^\perp,
\]
and one of the following conditions is fulfilled:

(a) \( M \) is \( D \)-minimal;
(b) \( M \) is \( D^\perp \)-minimal;
then \( M \) is CR totally geodesic.

The proof follows the same idea as in Theorem 3.

4. **Ricci tensor and scalar curvature of a CR manifold.** If \( \{E_1, \ldots, E_m\} \) is a local field of orthonormal frames on \( M \) such that \( \{E_{2p+1}, \ldots, E_{2p+q}\} \) is a local field of frames on \( D \) (resp. \( D^\perp \)), then by straightforward computation we have
\[
\sum_{i=1}^{m} \{ g(JP E_i, Y) g(JP X, E_i) \} = -g(PX, PY),
\]
\[
\sum_{i=1}^{m} \{ g(JP E_i, E_i) \} = 0,
\]
\[
\sum_{i=1}^{m} \{ g(E_i, JP X) g(E_i, JP Y) \} = g(PX, PY)
\]
for any vector fields \( X, Y \) on \( M \). If one uses (4.1)–(4.3), one establishes the following expression for the Ricci tensor of \( M \):
\[
S(X, Y) = \frac{m + 2}{4} \cdot cg(PX, PY) + \frac{m - 1}{4} \cdot cg(QX, QY)
+ \sum_{i=1}^{m} \{ g(h(X, Y), h(E_i, E_i)) - g(h(E_i, Y), h(E_i, X)) \}.
\]
In this way the scalar curvature of \( M \) is given by
\[
\rho = \frac{m^2 - m + 6p}{4} c
\]
(4.5)
\[
+ \sum_{i,j=1}^{m} \left\{ g(h(E_i, E_j), h(E_i, E_j)) - g(h(E_j, E_i), h(E_j, E_i)) \right\}.
\]
If \(\{\xi_1, \ldots, \xi_{2n-m}\}\) is a local basis of normal sections and \(A_a = A_{\xi_a}\), then we have
\[
h(X, Y) = \sum_{a=1}^{2n-m} g(A_aX, Y)\xi_a.
\]
(4.6)
Thus, (4.4) and (4.5) become
\[
S(X, Y) = \frac{m + 2}{4} \text{cg}(PX, PY) + \frac{m - 1}{4} \text{cg}(QX, QY)
\]
(4.7)
\[
+ \sum_{a=1}^{2n-m} \left\{ (\text{tr} A_a) g(A_aX, Y) - g(A_aX, A_aY) \right\},
\]
(4.8)
\[
\rho = \frac{m^2 - m + 6p}{4} c + \sum_{a=1}^{2n-m} (\text{tr} A_a)^2 - \|h\|^2.
\]
Therefore we have

**Theorem 5.** Let \(M\) be a minimal CR submanifold of the complex space form \(N(c)\). Then

(a) \[
S = \frac{m + 2}{4} \text{cg} \cdot (P \times P) - \frac{m - 1}{4} \text{cg} \cdot (Q \times Q)
\]
is negative semidefinite.

(b) \(\rho < ((m^2 - m + 6p)/4)c\).

Also, the following two theorems on totally geodesic CR submanifolds can be easily proved.

**Theorem 6.** A minimal CR submanifold \(M\) of a complex space form \(N(c)\) is totally geodesic, if and only if, one of the following conditions is satisfied:

(a) \[
S = \frac{m + 2}{4} \text{cg} \cdot (P \times P) + \frac{m - 1}{4} \text{cg} \cdot (Q \times Q),
\]

(b) \(\rho = ((m^2 - m + 6p)/4)c\).

**Theorem 7.** A CR submanifold \(M\) of a complex space form \(N(c)\) is totally geodesic, if and only if:

1. \(M\) is \(D^\perp\)-minimal.
2. The horizontal distribution \(D\) is involutive.
3. \(H(X) = c\) for any vector field \(X \in D\).
4. \(K_M(X \wedge Y) = c/4\) for any two vector fields \(X, Y\) on \(M\) such that \(Y \in D^\perp\).
5. Totally umbilical CR submanifolds. Suppose $M$ is totally umbilical, that is,

\begin{equation}
      h(X, Y) = g(X, Y)L,
\end{equation}

where $L$ is a normal vector field. Then we have

**Theorem 8.** If $M$ is a totally umbilical CR submanifold of a Kaehler manifold $N$, $m + q = 2n$, $m > 3$, and the horizontal distribution $D$ is parallel, then $M$ is totally geodesic.

**Proof.** From (2.10) and (2.11) we have

\begin{equation}
      g(J\nabla_x Y - A_{JQ}X, Z)
\end{equation}

for arbitrary vector fields on $M$. Using (5.2) and the fact that $g(h(X, Y), \xi) = g(A_\xi X, Y)$, we get the following relation:

\begin{equation}
      g(J\nabla_x Y, Z) - g(L, JQ)g(X, Z)
\end{equation}

Substitute $Y$ by $BL$ and $Z$ by $X$ and obtain

\begin{equation}
      g(X, X)g(L, JBL) + g(J\nabla_x BL, X) + g(AL, X)g(X, BL) + g(X, BL)^2 = 0.
\end{equation}

Now, choose $X$ as a unit vector field on $D$ (i.e. $X \in D_x$). Hence $g(X, BL) = g(JX, BL) = 0$. Differentiating the last relation we have $g(\nabla_x JX, BL) + g(JX, \nabla_x BL) = 0$. Since $D$ is parallel $g(\nabla_x JX, BL) = 0$, hence $g(X, J\nabla_x BL) = 0$. Then from (5.3) we have

\begin{equation}
      0 = g(L, JBL) = -g(JL, BL) = -g(BL, BL),
\end{equation}

hence $BL = 0$. Since $q = 2n - m$ we have $JL \in D^\perp$ which implies $L = 0$ and the proof is done.

**Remark.** Theorem 8 has been proved by G. D. Ludden, M. Okumura and K. Yano [3] for the particular case of totally real submanifolds of a complex manifold.

**Theorem 9.** A totally umbilical CR submanifold $M$ of a complex space form $N(c)$ is a space of constant curvature, if and only if, $N$ is a flat complex space.

**Proof.** For any plane of the tangent space of a totally umbilical CR submanifold is either

\begin{equation}
      K_M = c/4 + \|L\|^2
\end{equation}

or

\begin{equation}
      H = c + \|L\|^2.
\end{equation}

Then the theorem follows from (5.4) and (5.5).

**Theorem 10.** If $M$ ($m > 2$) is a totally umbilical compact CR submanifold of a hyperbolic complex space form $N(c)$ and if the CR sectional curvature of $M$ is negative, then the group of isometries of $M$ is finite.
Proof. Using in (4.4) the hypotheses of the theorem, we have
\begin{equation}
S(X, Y) = \left((m + 2)/4c + (m - 1)||L||^2\right)g(PX, PY) + (m - 1)(c/4 + ||L||^2)g(QX, QY).
\end{equation}
From (5.4) follows $c/4 + ||L||^2 < 0$ and $((m + 2)/4)c + (m - 1)||L||^2 < 0$. Hence the Ricci tensor given by (5.6) is negative definite. $M$ is compact, therefore the theorem follows from [2, Corollary 5.4, p. 251].

Remark. From (5.6) we see that Ricci tensor of a totally umbilical CR submanifold of an elliptic or flat complex space form is always positive definite.

In a forthcoming paper we shall given pinching theorems for CR submanifolds.

References

5. K. Yano, On a structure defined by a tensor field $f$ of type (1,1) satisfying $f^3 + f = 0$, Tensor 14 (1963), 99–109. MR 28 #2513.

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