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CR SUBMANIFOLDS OF A KAEHLER MANIFOLD. I

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ABSTRACT. The differential geometry of *CR* submanifolds of a Kaehler manifold is studied. Theorems about totally geodesic *CR* submanifolds and totally umbilical *CR* submanifolds are given.

1. Introduction. Many papers have been concerned with complex submanifolds of complex manifolds, especially of complex space forms (see [4] for a survey of results). Recently, B.Y. Chen and K. Ogiue [1] have studied totally real submanifolds of complex manifolds. Later these submanifolds were further investigated by K. Yano, M. Kon, G. D. Ludden and M. Okumura [3], [6], [7].

The purpose of this paper is to initiate a study of a new class of submanifolds of a complex manifold. In §2 we introduce the concept of *CR* submanifold and we give its basic properties. *CR* submanifolds have been studied, till now, only from the analytic viewpoint (i.e. concerning the complex structure). Different kinds of sectional curvature, Ricci tensor and scalar curvature of a *CR* submanifold of a complex space form are examined in §§3 and 4. Also, some results on totally geodesic *CR* submanifolds and totally umbilical *CR* submanifolds are proved.

2. *CR* submanifolds. Let N be a Kaehler manifold of complex dimension n and M be an m-dimensional Riemannian submanifold immersed in N. Denote by g (resp. g_0) the Kaehlerian metric on N (resp. the Riemannian metric on M), by J the almost complex structure on N and by φ the isometric immersion of M into N.

In order to simplify the presentation, we identify, for each $x \in M$, the tangent space T_xM with $\varphi_*(T_xM) \subset T_{\varphi(x)}N$ by means of φ. The Riemannian metric g_0 is identified with the restriction of g to the subspace $\varphi_*(T_xM)$. With this identification in mind we drop the sumbol g_0, using instead the symbol g.

Now, suppose on M a differentiable distribution D: $x \to D_x \subset T_xM$ ($\dim D_x = 2p$) is given. This distribution is assumed to be consistent with the almost complex structure on N, that is, $J(D_x) = D_x$ for each $x \in M$. Moreover, the complementary orthogonal distribution $D^\perp$: $x \to D_x^\perp \subset T_xM$

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($\dim D_x^\perp = q$) is supposed to be totally real, that is, $J(D_x^\perp) \subset \nu_x$ for each $x \in M$, where ν_x is the normal space to M at x.

The distribution D (resp. $D^\perp$) can be defined by a projector P (resp. Q) which satisfy the well-known conditions

$$P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad g \circ (P \times Q) = 0. \tag{2.1}$$

We call the distribution D (resp. $D^\perp$) the horizontal (resp. vertical) distribution on M.

Definition. The submanifold M endowed with the above pair of distributions $(D, D^\perp)$ is called a CR submanifold of N.

Remarks. 1. Any real curve or real hypersurface of N is automatically a CR submanifold.

2. If, in particular, $\dim D_x^\perp = 0$ (resp. $\dim D_x = 0$) for any $x \in M$, the CR submanifold M is a complex submanifold (resp. totally real submanifold) of N.

If ξ is a vector field in the normal bundle, put

$$J\xi = A\xi + B\xi + C\xi \tag{2.2}$$

where $A\xi$ (resp. $B\xi$) is the horizontal (resp. vertical) part of $J\xi$ and $C\xi$ the normal part. Thus, A (resp. B) is a horizontal (resp. vertical) valued 1-form on the normal bundle and C is an endomorphism of the normal bundle.

If X is a vector field on M, then JQX is a section in the normal bundle of M, and from (2.2) we have

$$BJQX + QX = 0, \tag{2.3}$$

$$AJQX = CJQX = 0. \tag{2.4}$$

Applying J to (2.2) and comparing horizontal, vertical and normal parts we obtain

$$C^2\xi + JB\xi + \xi = 0, \tag{2.5}$$

$$JA\xi + AC\xi = 0, \tag{2.6}$$

$$BC\xi = 0. \tag{2.7}$$

From (2.4) and (2.5) we get $C^3 + C = 0$ on the normal bundle, that is the structure introduced by K. Yano [5].

Let $\boldsymbol{\nabla}$ be the Kaehlerian connection on N. The Gauss and Weingarten equations are

$$\boldsymbol{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.8}$$

$$\boldsymbol{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi, \tag{2.9}$$

where ∇ is the Riemannian connection on M, $\nabla^\perp$ is the connection on the normal bundle induced by $\boldsymbol{\nabla}$ and h is the second fundamental form of the immersion. A_ξ is an endomorphism of the tangent bundle of M, and cannot be confused with the 1-form A defined by (2.2).

Since $\boldsymbol{\nabla}$ is a Kaehlerian connection, we have

$$P(\nabla_X JPY) - P(A_{JQY}X) = JP(\nabla_X Y) + Ah(X, Y), \tag{2.10}$$

$$Q(\nabla_X JPY) - Q(A_{JQY}X) = Bh(X, Y), \tag{2.11}$$

$$h(X, JPY) + \nabla^{\perp}_X JQY = JQ(\nabla_X Y) + Ch(X, Y) \tag{2.12}$$

for all vector fields X, Y on M.

Differentiating (2.2) and comparing horizontal, vertical and normal parts we obtain

$$P(\nabla_X A\xi) + P(\nabla_X B\xi) + JP(A_\xi X) = P(A_{C\xi}X) + A(\nabla^{\perp}_X \xi), \tag{2.13}$$

$$Q(\nabla_X A\xi) + Q(\nabla_X B\xi) = Q(A_{C\xi}X) + B(\nabla^{\perp}_X \xi), \tag{2.14}$$

$$h(X, A\xi) + h(X, B\xi) + \nabla^{\perp}_X C\xi + J(QA_\xi X) = C(\nabla^{\perp}_X \xi) \tag{2.15}$$

for each vector field X on M and normal section ξ.

3. Sectional curvature of a *CR* submanifold. Suppose now that N is a complex space form of constant holomorphic curvature c. Then, the curvature tensor $\mathbf{R}$ of $N(c)$ is given by

$$\begin{aligned}\mathbf{R}(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ &- g(JX, Z)JY + 2g(X, JY)JZ\}.\end{aligned} \tag{3.1}$$

The equation of Gauss becomes

$$\begin{aligned} &g(R(X, Y)Z, W) \\ &\quad = \frac{c}{4}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ &\qquad + g(JPY, Z)g(JPX, W) \\ &\qquad - g(JPX, Z)g(JPY, W) + 2g(X, JPY)g(JPZ, W)\} \\ &\qquad + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)).\end{aligned} \tag{3.2}$$

The sectional curvature K_M of M determined by orthonormal vectors X and Y is given by

$$\begin{aligned} K_M(X \wedge Y) = \frac{c}{4}\{1 + 3g(PX, JPY)^2\} &+ g(h(X, X), h(Y, Y)) \\ &- g(h(X, Y), h(X, Y)).\end{aligned} \tag{3.3}$$

Definition. The holomorphic sectional curvature H of M determined by a unit vector $X \in D$ is the sectional curvature determined by $\{X, JX\}$.

Hence from (3.3) we have

$$H(X) = c + g(h(X, X), h(JX, JX)) - g(h(X, JX), h(X, JX)). \tag{3.4}$$

From (2.12) we have

$$h(X, JY) = JQ(\nabla_X Y) + Ch(X, Y) \tag{3.5}$$

for any two vector fields on M which lie in D (i.e. X_x, $Y_x \in D_x$, $\forall x \in M$). As a consequence of (3.5) and (2.4) we obtain

$$h(JX, JY) = JQ(\nabla_{JX} Y) + C^2 h(X, Y) \quad \forall X, Y \in D. \tag{3.6}$$

Therefore, the holomorphic sectional curvature H of M determined by a unit vector $X \in D$ is given by

$$\begin{aligned} H(X) = {} & c + \|Bh(X,X)\|^2 - \|h(X,X)\|^2 - \|Ch(X,X)\|^2 \\ & - \|Q\,\nabla_X X\|^2 + g(h(X,X), JQ\,\nabla_{JX} X) \\ & - 2g(Ch(X,X), JQ\,\nabla_X X). \end{aligned} \tag{3.7}$$

REMARK. If, in particular, M is a complex submanifold of N (i.e. $Q = 0$, $J = C$), then we get the known formula for its holomorphic sectional curvature [4]:

$$H(X) = c - 2\|h(X,X)\|^2.$$

DEFINITION. The horizontal distribution D is called parallel with respect to the Riemannian connection ∇ on M if $\nabla_X Y \in D$ for any two vector fields X, $Y \in D$.

THEOREM 1. *If M is a CR submanifold of a complex space form $N(c)$ and D is an involutive distribution, then the holomorphic sectional curvature verifies $H(X) \leqslant c\ \forall X \in D$.*

PROOF. Since ∇ is the Levi-Civita connection on M and D is involutive, from (3.5) we have

$$h(X, JY) - h(JX, Y) = JQ(\nabla_X Y - \nabla_Y X) = JQ([X, Y]) = 0.$$

Hence $h(JX, JY) = -h(X, Y)$, and the assertion follows from (3.4).

REMARKS. 1. The distribution D is involutive, if and only if, $h(X, JY) = h(JX, Y)\ \forall X, Y \in D$.

2. If D is parallel with respect to ∇, then it is involutive and Theorem 1 is also valid.

DEFINITION. The CR submanifold M is called D-totally geodesic (resp. $D^{\perp}$-totally geodesic) if $h(X, Y) = 0$ for each $X, Y \in D$ (resp. $X, Y \in D^{\perp}$).

Then from (3.4) we have

THEOREM 2. *A CR submanifold M of a complex space form $N(c)$ is D-totally geodesic if and only if the following conditions are fulfilled:*

1. *The horizontal distribution is involutive.*
2. *$H(X) = c$ for each $X \in D$.*

Now, let $\{E_1, \ldots, E_m\}$ be a local field of orthonormal frames on M such that $\{E_1, \ldots, E_p, E_{p+1} = JE_1, \ldots, E_{2p} = JE_p\}$ (resp. $\{E_{2p+1}, \ldots, E_{2p+q}\}$) is a local field of frames in D (resp. $D^{\perp}$).

DEFINITION. The CR submanifold M is called D-minimal (resp. $D^{\perp}$-minimal, if $\Sigma_{i=1}^{2p}\{h(E_i, E_i)\} = 0$) (resp. $\Sigma_{i=1}^{q}\{h(E_{2p+i}, E_{2p+i})\} = 0$).

REMARK. Every CR submanifold with involutive horizontal distribution is D-minimal.

THEOREM 3. *If M is a $D^{\perp}$-minimal CR submanifold of a complex space form $N(c)$, then M is $D^{\perp}$-totally geodesic, if and only if,*

$$K_M(X \wedge Y) = c/4 \quad \forall X, Y \in D^{\perp}. \tag{3.8}$$

PROOF. Substitute X and Y from (3.3) by QX and QY and obtain

$$\begin{aligned} K_M(X \wedge Y) = c/4 &+ g(h(QX, QX), h(QY, QY)) \\ &- g(h(QX, QY), h(QX, QY)). \end{aligned} \tag{3.9}$$

Supposing (3.8) valid and taking into account the $D^{\perp}$-minimality of M, from (3.9) we have

$$g\big(h(E_{2p+i}, E_{2p+j}), h(E_{2p+i}, E_{2p+j})\big) = 0 \quad \forall 1 \leqslant i, j \leqslant q.$$

Therefore $h(X, Y) = 0$ for each vector field X, Y which lies in $D^{\perp}$. Of course, if M is $D^{\perp}$-totally geodesic, (3.8) follows from (3.9).

DEFINITION. A *CR* submanifold M is called *CR* totally geodesic, if $h(X, Y) = 0$ for any $X \in D$ and $Y \in D^{\perp}$. The sectional curvature determined by orthonormal vectors $X \in D$ and $Y \in D^{\perp}$ is called *CR* sectional curvature.

THEOREM 4. *If the CR sectional curvature of M is given by*

$$K_M(X \wedge Y) = c/4 \quad \forall X \in D, Y \in D^{\perp},$$

and one of the following conditions is fulfilled:

(a) *M is D-minimal*;

(b) *M is $D^{\perp}$-minimal*;

then M is CR totally geodesic.

The proof follows the same idea as in Theorem 3.

4. Ricci tensor and scalar curvature of a *CR* manifold. If $\{E_1, \dots, E_m\}$ is a local field of orthonormal frames on M such that $\{E_1, \dots, E_p, E_{p+1} = JE_1, \dots, E_{2p} = JE_p\}$ (resp. $\{E_{2p+1}, \dots, E_{2p+q}\}$) is a local field of frames on D (resp. $D^{\perp}$), then by straightforward computation we have

$$\sum_{i=1}^{m} \{ g(JPE_i, Y) g(JPX, E_i) \} = -g(PX, PY), \tag{4.1}$$

$$\sum_{i=1}^{m} \{ g(JPE_i, E_i) \} = 0, \tag{4.2}$$

$$\sum_{i=1}^{m} \{ g(E_i, JPX) g(E_i, JPY) \} = g(PX, PY) \tag{4.3}$$

for any vector fields X, Y on M. If one uses (4.1)–(4.3), one establishes the following expression for the Ricci tensor of M:

$$\begin{aligned} S(X, Y) = \frac{m+2}{4}\, cg(PX, PY) &+ \frac{m-1}{4}\, cg(QX, QY) \\ &+ \sum_{i=1}^{m} \{ g(h(X, Y), h(E_i, E_i)) - g(h(E_i, Y), h(E_i, X)) \}. \end{aligned} \tag{4.4}$$

In this way the scalar curvature of M is given by

$$\rho = \frac{m^2 - m + 6p}{4} c + \sum_{i,j=1}^{m} \{ g(h(E_j, E_j), h(E_i, E_i)) - g(h(E_i, E_j), h(E_i, E_j)) \}. \tag{4.5}$$

If $\{\xi_1, \ldots, \xi_{2n-m}\}$ is a local basis of normal sections and $A_\alpha = A_{\xi_\alpha}$, then we have

$$h(X, Y) = \sum_{\alpha=1}^{2n-m} g(A_\alpha X, Y)\xi_\alpha. \tag{4.6}$$

Thus, (4.4) and (4.5) become

$$S(X, Y) = \frac{m+2}{4} cg(PX, PY) + \frac{m-1}{4} cg(QX, QY) + \sum_{\alpha=1}^{2n-m} \{(\operatorname{tr} A_\alpha) g(A_\alpha X, Y) - g(A_\alpha X, A_\alpha Y)\}, \tag{4.7}$$

$$\rho = \frac{m^2 - m + 6p}{4} c + \sum_{\alpha=1}^{2n-m} (\operatorname{tr} A_\alpha)^2 - \|h\|^2. \tag{4.8}$$

Therefore we have

THEOREM 5. *Let M be a minimal CR submanifold of the complex space form $N(c)$. Then*

(a)

$$S - \frac{m+2}{4} cg \circ (P \times P) - \frac{m-1}{4} cg \circ (Q \times Q)$$

is negative semidefinite.

(b) $\rho \leq ((m^2 - m + 6p)/4)c$.

Also, the following two theorems on totally geodesic CR submanifolds can be easily proved.

THEOREM 6. *A minimal CR submanifold M of a complex space form $N(c)$ is totally geodesic, if and only if, one of the following conditions is satisfied:*

(a)

$$S = \frac{m+2}{4} cg \circ (P \times P) + \frac{m-1}{4} cg \circ (Q \times Q),$$

(b) $\rho = ((m^2 - m + 6p)/4)c$.

THEOREM 7. *A CR submanifold M of a complex space form $N(c)$ is totally geodesic, if and only if:*

1. *M is $D^\perp$-minimal.*
2. *The horizontal distribution D is involutive.*
3. *$H(X) = c$ for any vector field $X \in D$.*
4. *$K_M(X \wedge Y) = c/4$ for any two vector fields X, Y on M such that $Y \in D^\perp$.*

5. Totally umbilical *CR* submanifolds. Suppose M is totally umbilical, that is,

$$h(X, Y) = g(X, Y)L, \tag{5.1}$$

where L is a normal vector field. Then we have

THEOREM 8. *If M is a totally umbilical CR submanifold of a Kaehler manifold N, $m + q = 2n$, $m \geqslant 3$, and the horizontal distribution D is parallel, then M is totally geodesic.*

PROOF. From (2.10) and (2.11) we have

$$\begin{aligned} &g(\nabla_X JPY - A_{JQY}X, Z) \\ &\qquad = g(JP\,\nabla_X Y + Ah(X, Y) + Bh(X, Y), Z) \end{aligned} \tag{5.2}$$

for arbitrary vector fields on M. Using (5.2) and the fact that $g(h(X, Y), \xi) = g(A_\xi X, Y)$, we get the following relation:

$$\begin{aligned} &g(\nabla_X JPY, Z) - g(L, JQY)g(X,Z) \\ &\qquad = g(JP\,\nabla_X Y, Z) + g(AL, Z)g(X, Y) + g(BL, Z)g(X, Y). \end{aligned}$$

Substitute Y by BL and Z by X and obtain

$$\begin{aligned} &g(X, X)g(L, JBL) + g(JP\,\nabla_X BL, X) \\ &\qquad + g(AL, X)g(X, BL) + g(X, BL)^2 = 0. \end{aligned} \tag{5.3}$$

Now, choose X as a unit vector field on D (i.e. $X_x \in D_x$). Hence $g(X, BL) = g(JX, BL) = 0$. Differentiating the last relation we have $g(\nabla_X JX, BL) + g(JX, \nabla_X BL) = 0$. Since D is parallel $g(\nabla_X JX, BL) = 0$, hence $g(X, JP\,\nabla_X BL) = 0$. Then from (5.3) we have

$$0 = g(L, JBL) = -g(JL, BL) = -g(BL, BL),$$

hence $BL = 0$. Since $q = 2n - m$ we have $JL \in D^\perp$ which implies $L = 0$ and the proof is done.

REMARK. Theorem 8 has been proved by G. D. Ludden, M. Okumura and K. Yano [3] for the particular case of totally real submanifolds of a complex manifold.

THEOREM 9. *A totally umbilical CR submanifold M of a complex space form $N(c)$ is a space of constant curvature, if and only if, N is a flat complex space.*

PROOF. For any plane of the tangent space the sectional curvature of a totally umbilical *CR* submanifold is either

(5.4) $K_M = c/4 + \|L\|^2$ or

(5.5) $H = c + \|L\|^2$.

Then the theorem follows from (5.4) and (5.5).

THEOREM 10. *If M ($m \geqslant 2$) is a totally umbilical compact CR submanifold of a hyperbolic complex space form $N(c)$ and if the CR sectional curvature of M is negative, then the group of isometries of M is finite.*

PROOF. Using in (4.4) the hypotheses of the theorem, we have

$$S(X, Y) = \big((m+2)/4c + (m-1)\|L\|^2\big)g(PX, PY) + (m-1)\big(c/4 + \|L\|^2\big)g(QX, QY). \tag{5.6}$$

From (5.4) follows $c/4 + \|L\|^2 < 0$ and $((m+2)/4)c + (m-1)\|L\|^2 < 0$. Hence the Ricci tensor given by (5.6) is negative definite. M is compact, therefore the theorem follows from [**2**, Corollary 5.4, p. 251].

REMARK. From (5.6) we see that Ricci tensor of a totally umbilical CR submanifold of an elliptic or flat complex space form is always positive definite.

In a forthcoming paper we shall given pinching theorems for CR submanifolds.

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