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ANOTHER REALCOMPACT, 0-DIMENSIONAL, NON-N-COMPACT SPACE

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ABSTRACT. A refinement of the topology of the plane is constructed which is locally compact, locally countable, 0-dimensional, realcompact, but not N-compact.

P. Nyikos [5] exhibited an example of a topological space which was realcompact and 0-dimensional but not N-compact. Using a technique of E. K. van Douwen, we construct a locally compact refinement of the usual topology on the Euclidean plane which is realcompact and 0-dimensional but not N-compact.

1. Preliminaries. If E is a topological space, then a space X is called *E-compact* if X can be embedded as a closed subspace of some topological power of E. This concept was introduced by Engelking and Mrowka [3]. A space is called *0-dimensional* if it has a base of clopen (closed-and-open) sets. A space is called *realcompact* if it is R-compact, where R denotes the real line. Let N denote the set (and discrete space) of natural numbers. It is obvious that every N-compact space is realcompact and 0-dimensional. It had been an open question for some time whether or not every realcompact, 0-dimensional space was N-compact until Nyikos showed [5] that the space Δ constructed by Roy [6] is realcompact and 0-dimensional but not N-compact. The standard characterization of N-compact spaces that we require is the following result which can be found in [2].

1.1. THEOREM. *A 0-dimensional space X is N-compact if and only if every ultrafilter of clopen sets of X with the countable intersection property has nonempty intersection.*

We also require the following result which follows from the well-known fact that the Euclidean plane cannot be disconnected by the removal of a countable set.

1.2. LEMMA. *Let (X, T) be the plane with the usual metric topology. If U is an*

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open subset of X with a countable boundary, then either U is empty or $X - U$ is countable.

2. **The example.** E. K. van Douwen exhibits in [7] a technique for refining the topology of the real line to obtain (among other things) a locally compact, locally countable space S such that given two subsets A, B of the real line, if $\mathrm{cl}_R A \cap \mathrm{cl}_R B$ is uncountable then so is $\mathrm{cl}_S A \cap \mathrm{cl}_S B$. It is easy to verify that the same technique may be applied to the plane (X, T) with the usual metric topology to obtain a refinement (X, P) which satisfies the above conditions. The space (X, P) is our example.

Since P is a refinement of T, (X, P) is Hausdorff. Thus since (X, P) is locally compact, it must be completely regular. However, a completely regular, locally countable space is clearly 0-dimensional, hence (X, P) is 0-dimensional.

It is well known that the plane (X, T) is hereditarily realcompact. Theorem 8.17 of [4] states that if Y is hereditarily realcompact then any (completely regular and Hausdorff) space of which Y is a continuous, one-to-one image is realcompact. Thus (X, P) is realcompact. We show that (X, P) is not N-compact.

Let U be a clopen subset of (X, P). Then U and $X - U$ are clopen subsets of (X, P) and hence $\mathrm{cl}_{(X,P)} U \cap \mathrm{cl}_{(X,P)}(X - U) = \varnothing$. Thus by the condition mentioned above, we must have that $\mathrm{cl}_{(X,T)} U \cap \mathrm{cl}_{(X,T)}(X - U)$ is countable. Let $V = \mathrm{cl}_{(X,T)} U$ and let $W = \mathrm{cl}_{(X,T)}(X - U)$. Then $V \cap W$ is countable. Now, $X - V$ is open in (X, T), the plane. Clearly $X - V \subseteq W$, hence $\mathrm{cl}_{(X,T)}(X - V) \subseteq W$. Thus $\mathrm{cl}_{(X,T)} V \cap \mathrm{cl}_{(X,T)}(X - V) \subseteq V \cap W$, which is countable and, therefore, the boundary of $X - V$ in the space (X, T) is countable. By Lemma 1.2 either $X - V$ is empty or V is countable. If V is countable then since $U \subseteq V$, U is countable. If $X - V$ is empty then $V = X$ and hence $W = W \cap X = W \cap V$ is countable. But if W is countable then since $X - U \subseteq W$, $X - U$ must be countable. Thus either U or $X - U$ is countable if U is a clopen subset of (X, P). Let F denote the family of all clopen subsets of (X, P) with countable complements. Since every clopen subset of (X, P) is countable or has countable complement, F must be an ultrafilter. Clearly F has the countable intersection property. Since (X, P) is locally countable, F has empty intersection (every point of (X, P) has a countable clopen neighborhood). By Theorem 1.1 (X, P) is not N-compact.

3. **A remark.** The author showed in [1] that a locally compact, realcompact, 0-dimensional, weakly homogeneous space is N-compact and asked if weakly homogeneous or locally compact can be dropped from the hypotheses. The above example shows that weak homogeneity (a density condition on the Stone-Čech compactification) cannot be dropped from the hypotheses.

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