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ON THE STABLE COHOMOTOPY OF RP^∞

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ABSTRACT. There is a conjecture of G. B. Segal concerning the relation between the Burnside ring of G and the stable cohomotopy of BG. When $G = Z/2$ this conjecture is shown to be equivalent to the triviality of the group of homotopy classes of RP^∞ into the "cokernel of J".

1.1. **Introduction**. The Barratt-Priddy-Quillen theorem gives rise to a well-known homomorphism from the Burnside ring of a finite group, $\Omega(G)$, to the stable cohomotopy of its classifying space, $\pi_S^{\circ}(BG)$. Details of $\Omega(G)$ may be found in [3]. This homomorphism extends to the $I\Omega(G)$-adic completion of $\Omega(G)$ to give

$$\phi(G)\colon \Omega(G)^{\wedge} \to \pi_S^{\circ}(BG). \tag{1.1}$$

Around 1970 Graeme Segal conjectured, by analogy with K-theory [2], that $\phi(G)$ was an isomorphism. In this note I will relate (Theorem 1.6) Segal's conjecture when $G = Z/2$ to the following conjecture concerning $[RP^\infty, \operatorname{Cok} J]$, the set of homotopy classes of maps from RP^∞ to the "cokernel of J".

1.2. *Conjecture*. $[RP^\infty, \operatorname{Cok} J] = 0$.

In Remark 2.1 I will outline evidence in favour of the conjecture and give an equivalent reformulation (Conjecture 2.2). Firstly we must recall a few facts about the "cokernel of J", $\operatorname{Cok} J$. Henceforth all spaces will be 2-localised. Let $QS^{\circ} = \text{ind lim } \Omega^n S^n$, let $Q_m S^{\circ}$ be the "degree m" component and set $SG = Q_1 S^{\circ}$.

1.3. $\operatorname{Cok} J$. From the solution of the Adams conjecture [6], [8] there is a commutative diagram

$$\begin{array}{ccccc} JO & \xrightarrow{\pi'} & BSO & \xrightarrow{\psi^3 - 1} & BSO \\ \beta\downarrow & & \alpha\downarrow & & \| \\ SG & \xrightarrow{\pi} & G/O & \rightarrow & BSO \\ f\downarrow & & \downarrow e & & \\ JO & \xrightarrow{\pi'} & BSO & & \end{array} \tag{1.4}$$

The composites $e \circ \alpha$ and $f \circ \beta$ may be arranged to be H-space equivalences. The maps e and f are H-maps [5]. The common fibre of e and f is the H-space

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Cok J. Diagram (1.4) induces splittings

(1.5) $$SG = JO \times \mathrm{Cok}\, J \quad \text{and} \quad G/O = BSO \times \mathrm{Cok}\, J$$

satisfying $\pi = \pi' \times 1$.

1.6. THEOREM. *$\phi(Z/2)$ is an isomorphism if and only if Conjecture* 1.2 *is true.*

PROOF. It is well known that $\Omega(Z/2)^\wedge \cong Z \oplus \hat{Z}_2 \cong K^\circ(BZ/2)$ [2]. If d: $\pi_S^\circ(BZ/2) \to K^\circ(BZ/2)$ is the d-invariant for unitary K-theory it is easy to see that $d \circ \phi(Z/2)$ is an isomorphism. Hence if $[RP^\infty, \mathrm{Cok}\, J] = 0$ then

$$\begin{aligned} \pi_S^\circ(BZ/2) &= [BZ/2, QS^\circ] \quad \text{(unbased homotopy classes of maps)} \\ &\cong Z \oplus [RP^\infty, SG] \\ &\cong Z \oplus [RP^\infty, JO] \\ &\cong Z \oplus \widetilde{KO}^\circ(RP^\infty) \quad \text{(easily deduced from (1.4))} \\ &\cong K^\circ(RP^\infty) \text{ (see [2])} \\ &\cong \Omega(Z/2)^\wedge. \end{aligned}$$

Conversely if $0 \neq h$: $RP^\infty \to \mathrm{Cok}\, J$ then composition of h with the inclusion of Cok J into SG gives a nonzero element$x \in \pi_S^\circ(BZ/2)$. However $d(x) = 0$ since $\tilde{K}^\circ(\mathrm{Cok}\, J) = 0$ [4], [7, 9.9].

2.1. **Remark.** It is difficult to construct elements in $[RP^\infty, \mathrm{Cok}\, J]$. My leading candidate for nontriviality is a composition of the form

$$RP^\infty \xrightarrow{\Delta} BO \times BO \xrightarrow{\delta} G/O \to \mathrm{Cok}\, J$$

where Δ is the diagonal, δ is the deviation from additivity of a solution of the Adams conjecture, $BO \to G/O$, and the third map is the projection obtained from (1.5). At one time I falsely claimed that this map was detected on the bottom dimensional S^6 in Cok J. Ib Madsen pointed out the error. Subsequent correspondence and computation left us convinced that the above map is not detected by its induced homomorphism on $H_*(-; Z/2)$ in any dimension less than or equal to eight. Several others familiar with SG and G/O were at first confident that they could produce a nonzero element. However, after much industry, this body of opinion is now unanimous in its belief of Conjecture 1.2.

There is other evidence. The set of stable homotopy classes $\{RP^\infty, RP^\infty\}$ is a 2-adic local ring. This follows from [9] and from the fact that such an S-map is a stable equivalence if and only if it induces the identity on $H_1(RP^\infty, Z/2)$. By the Kahn-Priddy theorem [1] there is an epimorphism

$$\{RP^\infty, RP^\infty\} \to \tilde{\pi}_S^\circ(RP^\infty)$$

so that the following conjecture implies conjecture 1.2.

2.2. *Conjecture.* $\{RP^\infty, RP^\infty\} \cong \hat{Z}_2$, the 2-adics, generated by the class of the identity map.

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