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THE PARALLELIZABILITY OF THE STIEFEL MANIFOLDS $V_k(\mathbf{R}^n)$: $k \geq 3$

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ABSTRACT. The purpose of this note is to prove the result stated in the title by constructing a Gauss map for the standard embedding $V_k(\mathbf{R}^n) \subset \mathbf{R}^{nk}$.

Let $V_k(\mathbf{R}^n)$ denote the Stiefel manifold of orthonormal k frames in $\mathbf{R}^n$. It has been shown by Sutherland [3] and later Lam [2] that $V_k(\mathbf{R}^n)$ is parallelizable for $k \geqslant 2$. Using the idea of the Gauss map of a framed immersion we give an extremely simple proof of this fact for $k \geqslant 3$, obtaining the explicit formula of Lam for the tangent bundle of $V_k(\mathbf{R}^n)$ for all k, from which the main result follows in a line. The proof works almost verbatim for $V_k(\mathbf{F}^n)$ $k \geqslant 3$ with $\mathbf{F} = \mathbf{C}$ or $\mathbf{H}$. However the details are best carried out for $\mathbf{R}$ alone.

GAUSS MAPS. Let f: $M^m \looparrowright \mathbf{R}^{m+k}$ be an immersion with normal bundle ν, and suppose the bundle isomorphism Φ: $\nu \xrightarrow{\sim} \mathbf{R}^k \downarrow M^m$ gives a trivialization of ν. Then at each point $x \in M^m$ we obtain a k frame $\gamma_f(x) \in V_k(\mathbf{R}^n)$ by taking the image under Φ^{-1} of the standard basis of $\mathbf{R}^k$, regarding it as a normal frame to $f(M) \subset \mathbf{R}^{m+k}$ at $f(x)$, and translating it down to the origin of $\mathbf{R}^{m+k}$. The map

$$\gamma_f\colon M^m \to V_k(\mathbf{R}^{m+k})$$

is called the Gauss map of the framed immersion (f: $M \looparrowright \mathbf{R}^{m+k}$; Φ). It is easily seen that γ_f is covered by a bundle map,

$$\Gamma_f\colon \tau(M)\downarrow M \to \eta\downarrow V_k(\mathbf{R}^{m+k})$$

that is iso on fibres, where $\eta\downarrow V_k(\mathbf{R}^{m+k})$ is the canonical m-plane bundle. (The fibre of η over a point $[v_1, \ldots, v_k] \in V_k(\mathbf{R}^{m+k})$ is the subspace of $\mathbf{R}^{m+k}$ orthogonal to the span of $\{v_1, \ldots, v_k\}$.) Therefore $\gamma_f^*(\eta) \simeq (\tau M)\downarrow M$. To construct a Gauss map for $V_k(\mathbf{R}^n)$ observe that by definition $V_k(\mathbf{R}^n) \subset \mathbf{R}^n \times \cdots \times \mathbf{R}^n = \mathbf{R}^{kn}$. Regard $\mathbf{R}^{k(k+1)/2}$ as the space of upper triangular $k \times k$ matrices and define ϕ: $\mathbf{R}^{kn} \to \mathbf{R}^{k(k+1)/2}$ by the formula

$$\phi(u_1, \ldots, u_k) = \begin{cases} \|u_i\|^2 - 1, & i = j, \\ \langle u_i | u_j \rangle, & i > j, \\ 0, & i < j, \end{cases}$$

where $u_r \in \mathbf{R}^n$, $r = 1, \ldots, k$. Then ϕ is transverse regular to $0 \in \mathbf{R}^{k(k+1)/2}$

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and (by definition!) $V_k(\mathbf{R}^n) = \phi^{-1}(0)$. Therefore the normal bundle $\nu(V_k(\mathbf{R}^n) \hookrightarrow \mathbf{R}^{kn})$ gets framed by pulling a frame at the origin $0 \in \mathbf{R}^{kn}$ back along the differential $d\phi$. Let

$$\gamma\colon V_k(\mathbf{R}^n) \to V_{k(k+1)/2}(\mathbf{R}^{nk})$$

be the Gauss map associated to this trivialization. Unraveling the definition one sees that γ is the composite

$$\begin{array}{ccc} V_k(\mathbf{R}^n) & \xrightarrow{\pi} & V_k(\mathbf{R}^n) \times V_{k-1}(\mathbf{R}^n) \times \cdots \times V_1(\mathbf{R}^n) \\ \downarrow\gamma & & \downarrow j \\ V_{k(k+1)/2}(\mathbf{R}^{nk}) & \equiv & V_{k+k-1+\cdots+1}(\mathbf{R}^n \times \cdots \times \mathbf{R}^n) \end{array}$$

where the ith component of π is $\pi_i\colon V_k(\mathbf{R}^n) \to V_{k-i+1}(\mathbf{R}^n)$ given by

$$\pi_i[v_1, \ldots, v_k] = [v_i, \ldots, v_k], \qquad i = 1, \ldots, k,$$

and j is the canonical map. Letting η denote the canonical bundle over whatever Stiefel manifold is relevant one has $\pi_{i+1}^*(\eta) = \mathbf{R}^i \oplus \eta$ over $V_k(\mathbf{R}^n)$. Since

$$j^*(\eta) = \eta \times \cdots \times \eta \downarrow \mathop{\times}_{i=1}^{k} V_{k-i+1}(\mathbf{R}^n)$$

we get $\gamma^*(\eta) = \bigoplus_{i=1}^k \mathbf{R}^{i-1} \oplus \eta$ and since γ is a Gauss map $\gamma^*(\eta) = \tau$, so

$$\tau(V_k(\mathbf{R}^n)) = \mathbf{R}^{k(k-1)/2} \oplus k\eta.$$

Recalling that $\eta \oplus \mathbf{R}^k \downarrow V_k(\mathbf{R}^n)$ is naturally iso to $\mathbf{R}^n \downarrow V_k(\mathbf{R}^n)$ one obtains for $k \geqslant 3$ that $\tau(V_k(\mathbf{R}^n))$ is trivial.

Remark. The definition of ϕ is implicit in [1, p. 197/4] and the formulae for $\tau(V_k(\mathbf{R}^n))$ comes from [2, p. 309/–9].

References

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