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# ON PROJECTIVE PRIME IDEALS IN $C(X)$

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ABSTRACT. This note presents characterizations of the projective prime ideals in $C(X)$ and of the hereditary and semihereditary rings of continuous functions.

This note presents a characterization of the projective prime ideals in $C(X)$, the ring of real-valued continuous functions on a completely regular Hausdorff space $X$. This characterization is then applied to obtain a characterization of the hereditary rings of continuous functions. The reader is referred to [5] and [2] for background. The referee has pointed out that the results in this paper can also be derived from the more general results appearing in [4].

LEMMA. *Each projective prime ideal in $C(X)$ is generated by an idempotent.*

PROOF. The following argument shows that a projective prime ideal is finitely generated. The lemma then follows from Theorem 3 of [3].

Suppose $P$ is a nonfinitely generated projective ideal in $C(X)$. By Theorem 2.4 of [2], $P$ is generated by a family $\{f_\alpha\}_{\alpha\in A}$ such that $\{\operatorname{coz} f_\alpha\}_{\alpha\in A}$ is star-finite. There is a countably infinite subset $\{Y_i\}_{i=1}^{\infty} \subseteq \{\operatorname{coz} f_\alpha\}_{\alpha\in A}$ such that $Y_i \cap Y_j = \varnothing$ if $i \neq j$. Now, by the complete regularity of $X$, for each $i$ there is a $g_i \in C(X)$ such that $0 \leqslant g_i \leqslant \frac{1}{2^i}$ and $\varnothing \neq \operatorname{coz} g_i \subseteq Y_i$. Neither $\sum_{i=1}^{\infty} g_{2i}$ nor $\sum_{i=1}^{\infty} g_{2i+1}$ can be a finite linear combination of elements of $\{f_\alpha\}_{\alpha\in A}$ since $\{\operatorname{coz} f_\alpha\}_{\alpha\in A}$ is star-finite. However, both sums are in $C(X)$ and their product is $0 \in P$. Thus, $P$ is not prime.

If $x \in X$, let $M_x$ denote the maximal ideal of $C(X)$ consisting of functions whose zero-sets contain $x$.

THEOREM 1. *A proper ideal in $C(X)$ is a projective prime ideal if and only if it has the form $M_x$ for some isolated $x \in X$.*

PROOF. If $x \in X$ is isolated, then $M_x$ is a summand of $C(X)$ and hence projective.

Suppose $P$ is a proper projective prime ideal in $C(X)$. By the previous lemma, $P$ is principal and hence fixed since it is proper. Thus, it is contained in a fixed maximal ideal. But $P$ is contained in only one maximal ideal [5, 2.11], so the idempotent generating $P$ must be the characteristic function of

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$X \setminus \{x\}$ for some $x \in X$. Thus, $P = M_x$ for some isolated $x \in X$.

In [1] Bergman presents characterizations for the hereditary and semihereditary commutative rings which can be applied to produce the following results. However, the application of the results in [2] and the preceding theorem provide a more straightforward and revealing approach in the restricted setting of $C(X)$.

First, it was shown in [2] that a principal ideal in $C(X)$ is projective if and only if the support of the generating function is open. Moreover, if $X$ is basically disconnected, then every finitely generated ideal in $C(X)$ is principal [5, 14N.4 and 14.25]. Thus, $C(X)$ is semihereditary if and only if $X$ is basically disconnected.

THEOREM 2. *The following are equivalent.*

(a) *$C(X)$ is hereditary.*

(b) *Every prime ideal is projective.*

(c) *Every maximal ideal is projective.*

(d) *$X$ is finite and discrete.*

PROOF. Clearly (a) implies (b) which implies (c). Moreover, if all maximal ideals are projective, they must be fixed at isolated points by Theorem 1. Thus, $X$ must be compact and discrete. Consequently, (c) implies (d). Finally, if $X$ is finite and discrete, every ideal in $C(X)$ is a summand so (d) implies (a).

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