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A TERNARY FUNCTION FOR DISTRIBUTIVITY AND PERMUTABILITY OF AN EQUIVALENCE LATTICE

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ABSTRACT. The main result of the paper is

THEOREM 1. *Let A be a countable set and L be a complete sublattice of the equivalence lattice on A. The following are equivalent*

(i) *L is a distributive lattice of permutable equivalence relations.*

(ii) *There is an algebra with congruence lattice L among the fundamental operations of which is a ternary function f with the property*

(1) $$f(a, b, b) = f(a, b, a) = f(b, b, a) = a$$

for all $a, b \in A$.

This theorem is a contribution to the concrete representation problem for congruence lattices. Other results related to this problem can be found in [2]. We always assume that a complete sublattice of a complete lattice U has the same extremal elements as U. Suppose ϑ is an equivalence relation on the set A and g is a function from some subset X of A^3 into A. We write $(a, b, c)\ \vartheta\ (d, e, f)$ for $a\ \vartheta\ d$, $b\ \vartheta\ e$, $c\ \vartheta\ f$; g is compatible with ϑ provided $g(a, b, c)\ \vartheta\ g(d, e, f)$ whenever $(a, b, c) \in X$, $(d, e, f) \in X$ and $(a, b, c)\ \vartheta\ (d, e, f)$; g is compatible with a set L of congruence relations if it is compatible with each member of L. Analogous definitions are used for unary functions. In the proof of Theorem 1 we use:

THEOREM 2. *Let L be a complete distributive lattice of permutable equivalence relations on the countable set A. There is a function f from A^3 into A which is compatible with L and for which* (1) *holds for all $a, b \in A$.*

An analogous theorem, for A arbitrary and L finite, was proved in [3] (L was considered as a lattice of congruences of some algebra; however this fact was not substantially used). At the Colloquium on Universal Algebra in Oberwolfach, July 1973, A. F. Pixley asked whether the finiteness of L can be omitted in his theorem. Our theorems give a partial answer to his question.

PROOF OF THEOREM 2. For the sake of convenience we suppose that A is countably infinite. The finite case may be obtained by halting our construction of f below at the appropriate place, but this case was fully established in [3] (cf. Lemma 3.1 where the assumption that L is the

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congruence lattice of some algebra is extraneous). We may also suppose $A^2 \in L$. Let

$$D = \{(a, b, c) \in A^3: a, b, c \text{ are pairwise different}\},$$

$F = A^3 - D$. Now D is countably infinite, so we impose on it the order type of the natural numbers: $t_0 < t_1 < t_2 \ldots$. Define

$$S(a, b, c) = \{(d, e, f): \{d, e, f\} \subsetneq \{a, b, c\}\}$$

and let

$$g(a, b, c) = \begin{cases} a & \text{if } b = c \\ c & \text{otherwise} \end{cases} \quad \text{for } (a, b, c) \in F.$$

It is easy to verify that g is compatible with all equivalence relations on A.

LEMMA 1. *If the domain of h_k consists of all $t < t_k$ with $t \in D$ and $h_k \cup g$ is compatible with L, then h_k can be extended to h_{k+1} with the domain $\{t_i: 0 \leqslant i \leqslant k\}$ so that $h_{k+1} \cup g$ is compatible with L.*

PROOF OF LEMMA 1. Let $f_k = h_k \cup g$ and $E_k = \{t_i: 0 \leqslant i < k\} \cup S(t_k)$. E_k is finite and E_k is a subset of the domain of f_k. For each $t \in E_k$ let ϑ_t be the least equivalence relation in L with $t\ \vartheta_t\ t_k$. Evidently $t\ \vartheta_t\vartheta_s\ s$ for all $t, s \in E_k$. Since $\vartheta_t\vartheta_s = \vartheta_t \vee \vartheta_s$, we have $\vartheta_t\vartheta_s \in L$, so by the compatibility of f_k we obtain $f_k(t)\ \vartheta_t\vartheta_s\ f_k(s)$. Consequently, by the Chinese Remainder Theorem (see Pixley [3] or Grätzer [1, p. 211, Exercise 68]) we conclude that there is $d \in A$ such that $f_k(t)\ \vartheta_t\ d$ for all $t \in E_k$. Let $h_{k+1}(t_k)$ be such a d. To see that f_{k+1} $(= h_{k+1} \cup g)$ is compatible with L suppose that $t \in E_k \cup F$ and $t\ \vartheta\ t_k$ for some $\vartheta \in L$. If $t \in E_k$ we have $\vartheta_t \subseteq \vartheta$ and $t\ \vartheta_t\ t_k$. But then $f_{k+1}(t) = f_k(t)\ \vartheta_t\ h_{k+1}(t_k) = f_{k+1}(t_k)$, and so $f_{k+1}(t)\ \vartheta\ f_{k+1}(t_k)$. If $t \in F$ we can suppose $t = (a, a, b)$; the other two cases are similar. Let $t_k = (x, y, z)$. Then $x\ \vartheta\ a\ \vartheta\ y$ and $z\ \vartheta\ b$. So $x\ \vartheta\ y$ and we obtain $(x, x, z)\ \vartheta\ t_k$. But $(x, x, z) \in S(t_k) \subseteq E_k$. Hence

$$f_{k+1}(t_k)\ \vartheta\ f_k(x, x, z) = z\ \vartheta\ b = f_{k+1}(a, a, b).$$

This completes the proof of Lemma 1.

By well ordering A we can define f by the following recursion using Lemma 1 to do the crucial step.

h_0 is the empty function;

d_k is the least $e \in A$ such that $h_k \cup \{(t_k, e)\} \cup g$ is compatible with L;

$h_{k+1} = h_k \cup \{(t_k, d_k)\}$.

Then let $f = g \cup \bigcup_{k=0}^{\infty} h_k$. Observe that (1) holds since $g \subseteq f$. Let $t, s \in A^3$ and suppose $t\ \vartheta\ s$ for some $\vartheta \in L$. There is a k such that t and s are both in the domain of $g \cup h_k$. Since $g \cup h_k$ is compatible with L we conclude

$$f(t) = (g \cup h_k)(t)\ \vartheta\ (g \cup h_k)(s) = f(s).$$

In this way Theorem 2 is established.

PROOF OF THEOREM 1. It suffices to prove that (i) implies (ii); the converse was proved in [3, p. 183]. If (i) holds, then by Theorem 2 there is a function f

from A^3 into A which is compatible with L and which satisfies (1) for all $x, z \in A$; it will be one of the fundamental operations of the algebra which we construct. The remainder fundamental operations of it will exclude the equivalence relations not belonging to L.

LEMMA 2. *If $\eta \notin L$ is an equivalence relation on A then there is a unary operation g on A which is compatible with L and not compatible with η.*

PROOF OF LEMMA 2. Let $\vartheta_{(x,y)}$ be the least element ξ of the lattice L such that $x \,\xi\, y$. Then obviously $\eta \leqslant \bigvee_{(a,b)\in\eta} \vartheta_{(a,b)}$. However, $\eta \notin L$ and thus the equality does not hold. Therefore there is $(a_0, a_1) \in \eta$ such that $\vartheta_{(a_0,a_1)} \nleqslant \eta$; hence there are c_0, c_1 such that $\neg\, c_0 \,\eta\, c_1$, $c_0 \,\vartheta_{(a_0,a_1)}\, c_1$. Take these a_0, a_1, c_0, c_1 and arbitrary $\xi \in L$. Then $a_0 \,\xi\, a_1$ implies $\xi \geqslant \vartheta_{(a_0,a_1)}$ and, hence, $c_0 \,\xi\, c_1$. Thus the unary partial function $g_2 = \{(a_0, c_0), (a_1, c_1)\}$ is compatible with L; it obviously is not compatible with η. (Up to this moment we have used neither countability of A nor (i).)

Impose on $A - \{a_0, a_1\}$ the order type of natural numbers $a_2 < a_3 < a_4 < \ldots$. Suppose that $k \geqslant 2$ and that $g_k = \{(a_0, c_0), (a_1, c_1), \ldots, (a_{k-1}, c_{k-1})\}$ is constructed, g_k is compatible with L. Let ϑ_i $(0 \leqslant i < k)$ be the least element of L such that $a_i \,\vartheta_i\, a_k$. In the same way as in Lemma 1 we can find the least j such that $c_i \,\vartheta_i\, a_j$ for all $i < k$, and show that $g_{k+1} = g_k \cup \{(a_k, c_k)\}$, where $c_k = a_j$, is compatible with L. Then $g = \bigcup_{k=2}^{\infty} g_k$ is also compatible with L. Since $a_0 \,\eta\, a_1$, $\neg\, g(a_0) \,\eta\, g(a_1)$, the function g is not compatible with η. Lemma 2 is established.

For every equivalence relation ϑ on A, $\vartheta \notin L$, let g_ϑ be a unary function compatible with L and not compatible with ϑ. Let the set of fundamental operations of an algebra $\mathcal{A}$ consist of f and all g_ϑ; their ordering is not important. Then L is the congruence lattice of $\mathcal{A}$. Q.E.D.

REMARKS. 1. After g_2 was constructed in the proof of Lemma 2 we have not used η in the construction of g. Hence we may ask that the algebra in (ii) has countable signature.

2. Using the first part of the proof of Lemma 2 we can show: If A is an infinite set, B is the set of all complete sublattices of the equivalence lattice on A, then $\operatorname{card}(B) = 2^{\operatorname{card}(A)}$.

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