A TERNARY FUNCTION FOR
DISTRIBUTIVITY AND PERMUTABILITY
OF AN EQUIVALENCE LATTICE

IVAN KOREC

Abstract. The main result of the paper is

Theorem 1. Let $A$ be a countable set and $L$ be a complete sublattice of the
equivalence lattice on $A$. The following are equivalent

(i) $L$ is a distributive lattice of permutable equivalence relations.
(ii) There is an algebra with congruence lattice $L$ among the fundamental
operations of which is a ternary function $f$ with the property

$\quad f(a, b, b) = f(a, b, a) = f(b, b, a) = a$

for all $a, b \in A$.

This theorem is a contribution to the concrete representation problem for
congruence lattices. Other results related to this problem can be found in [2].
We always assume that a complete sublattice of a complete lattice $U$ has the
same extremal elements as $U$. Suppose $\theta$ is an equivalence relation on the set
$A$ and $g$ is a function from some subset $X$ of $A^3$ into $A$. We write

$(a, b, c) \theta (d, e, f)$ for $a \theta d, b \theta e, c \theta f$; $g$ is compatible with $\theta$ provided
$g(a, b, c) \theta g(d, e, f)$ whenever $(a, b, c) \in X, (d, e, f) \in X$ and
$(a, b, c) \theta (d, e, f); g$ is compatible with a set $L$ of congruence relations if it is
compatible with each member of $L$. Analogous definitions are used for unary
functions. In the proof of Theorem 1 we use:

Theorem 2. Let $L$ be a complete distributive lattice of permutable equivalence
relations on the countable set $A$. There is a function $f$ from $A^3$ into $A$ which is
compatible with $L$ and for which (1) holds for all $a, b \in A$.

An analogous theorem, for $A$ arbitrary and $L$ finite, was proved in [3] ($L$
was considered as a lattice of congruences of some algebra; however this fact
was not substantially used). At the Colloquium on Universal Algebra in
Oberwolfach, July 1973, A. F. Pixley asked whether the finiteness of $L$ can be
omitted in his theorem. Our theorems give a partial answer to his question.

Proof of Theorem 2. For the sake of convenience we suppose that $A$ is
countably infinite. The finite case may be obtained by halting our
construction of $f$ below at the appropriate place, but this case was fully
established in [3] (cf. Lemma 3.1 where the assumption that $L$ is the
congruence lattice of some algebra is extraneous). We may also suppose $A^2 \in L$. Let

$$D = \{(a, b, c) \in A^3: a, b, c \text{ are pairwise different}\},$$

$F = A^3 - D$. Now $D$ is countably infinite, so we impose on it the order type of the natural numbers: $t_0 < t_1 < t_2 \ldots$. Define

$$S(a, b, c) = \{(d, e, f): (d, e, f) \subseteq \{a, b, c\}\}$$

and let

$$g(a, b, c) = \begin{cases} a & \text{if } b = c \\ c & \text{otherwise} \end{cases} \text{ for } (a, b, c) \in F.$$

It is easy to verify that $g$ is compatible with all equivalence relations on $A$.

**Lemma 1.** If the domain of $h_k$ consists of all $t < t_k$ with $t \in D$ and $h_k \cup g$ is compatible with $L$, then $h_k$ can be extended to $h_{k+1}$ with the domain $(t_i: 0 < i < k)$ so that $h_{k+1} \cup g$ is compatible with $L$.

**Proof of Lemma 1.** Let $f_k = h_k \cup g$ and $E_k = \{t_i: 0 < i < k\} \cup S(t_k)$. $E_k$ is finite and $E_k$ is a subset of the domain of $f_k$. For each $t \in E_k$ let $\vartheta_t$ be the least equivalence relation in $L$ with $t \vartheta_t t_k$. Evidently $t \vartheta_s \vartheta_t$ for all $s, t \in E_k$. Since $\vartheta_t = \vartheta_s \lor \vartheta_t$, we have $\vartheta_t \vartheta_t \in L$, so by the compatibility of $f_k$ we obtain $f_k(t) \vartheta_t f_k(s)$. Consequently, by the Chinese Remainder Theorem (see Pixley [3] or Grätzer [1, p. 211, Exercise 68]) we conclude that there is $d \in A$ such that $f_k(t) d, d$ for all $t \in E_k$. Let $h_{k+1}(t_k)$ be such a $d$. To see that $f_{k+1}$ is compatible with $L$ suppose that $t \in E_k \cup F$ and $t \vartheta t_k$ for some $\vartheta \in L$. If $t \in E_k$ we have $\vartheta_t \vartheta_t \subseteq \vartheta_t$ and $t \vartheta_t t_k$. But then $f_{k+1}(t) = f_k(t) \vartheta_t f_{k+1}(t)$, and so $f_{k+1}(t) \vartheta f_{k+1}(t)$. If $t \in F$ we can suppose $t = (a, a, b)$; the other two cases are similar. Let $t_k = (x, y, z)$. Then $x \vartheta a \vartheta y$ and $z \vartheta b$. So $x \vartheta y$ and we obtain $(x, x, z) \vartheta t_k$. But $(x, x, z) \in S(t_k) \subseteq E_k$. Hence

$$f_{k+1}(t_k) \vartheta f_k(x, x, z) = z \vartheta b = f_{k+1}(a, a, b).$$

This completes the proof of Lemma 1.

By well ordering $A$ we can define $f$ by the following recursion using Lemma 1 to do the crucial step.

$h_0$ is the empty function;

$d_k$ is the least $e \in A$ such that $h_k \cup \{(t_k, e)\} \cup g$ is compatible with $L$;

$h_{k+1} = h_k \cup \{(t_k, d_k)\}$.

Then let $f = g \cup \bigcup_{k=0}^\infty h_k$. Observe that (1) holds since $g \subseteq f$. Let $t, s \in A^3$ and suppose $t \vartheta s$ for some $\vartheta \in L$. There is a $k$ such that $t$ and $s$ are both in the domain of $g \cup h_k$. Since $g \cup h_k$ is compatible with $L$ we conclude

$$f(t) = (g \cup h_k)(t) \vartheta (g \cup h_k)(s) = f(s).$$

In this way Theorem 2 is established.

**Proof of Theorem 1.** It suffices to prove that (i) implies (ii); the converse was proved in [3, p. 183]. If (i) holds, then by Theorem 2 there is a function $f$
from $A^3$ into $A$ which is compatible with $L$ and which satisfies (1) for all $x, z \in A$; it will be one of the fundamental operations of the algebra which we construct. The remainder fundamental operations of it will exclude the equivalence relations not belonging to $L$.

**Lemma 2.** If $\eta \notin L$ is an equivalence relation on $A$ then there is a unary operation $g$ on $A$ which is compatible with $L$ and not compatible with $\eta$.

**Proof of Lemma 2.** Let $\vartheta_{(x,y)}$ be the least element $\xi$ of the lattice $L$ such that $x \leq \xi y$. Then obviously $\eta \leq \bigwedge_{(a,b) \in \eta} \vartheta_{(a,b)}$. However, $\eta \notin L$ and thus the equality does not hold. Therefore there is $(a_0, a_1) \in \eta$ such that $\vartheta_{(a_0,a_1)} \notin \eta$; hence there are $c_0, c_1$ such that $\nvert c_0 \eta c_1$, $c_0 \vartheta_{(a_0,a_1)} c_1$. Take these $a_0, a_1, c_0, c_1$ and arbitrary $\xi \in L$. Then $a_0 \xi a_1$ implies $\xi \geq \vartheta_{(a_0,a_1)}$ and, hence, $c_0 \xi c_1$. Thus the unary partial function $g_2 = \{(a_0, c_0), (a_1, c_1)\}$ is compatible with $L$; it obviously is not compatible with $\eta$. (Up to this moment we have used neither countability of $A$ nor (i).)

Impose on $A - \{a_0, a_1\}$ the order type of natural numbers $a_2 < a_3 < a_4 < \ldots$. Suppose that $k \geq 2$ and that $g_k = \{(a_0, c_0), (a_1, c_1), \ldots, (a_{k-1}, c_{k-1})\}$ is constructed, $g_k$ is compatible with $L$. Let $\vartheta_i$ ($0 < i < k$) be the least element of $L$ such that $a_i \vartheta_i a_k$. In the same way as in Lemma 1 we can find the least $j$ such that $c_i \vartheta_i a_j$ for all $i < k$, and show that $g_{k+1} = g_k \cup \{(a_k, c_k)\}$, where $c_k = a_j$, is compatible with $L$. Then $g = \bigcup_{k=2}^\infty g_k$ is also compatible with $L$. Since $a_2 \eta a_1, \nvert g(a_0) \eta g(a_1)$, the function $g$ is not compatible with $\eta$. Lemma 2 is established.

For every equivalence relation $\vartheta$ on $A$, $\vartheta \notin L$, let $g_\vartheta$ be a unary function compatible with $L$ and not compatible with $\vartheta$. Let the set of fundamental operations of an algebra $\mathcal{A}$ consist of $f$ and all $g_\vartheta$; their ordering is not important. Then $L$ is the congruence lattice of $\mathcal{A}$. Q.E.D.

**Remarks.** 1. After $g_2$ was constructed in the proof of Lemma 2 we have not used $\eta$ in the construction of $g$. Hence we may ask that the algebra in (ii) has countable signature.

2. Using the first part of the proof of Lemma 2 we can show: If $A$ is an infinite set, $B$ is the set of all complete sublattices of the equivalence lattice on $A$, then $\operatorname{card}(B) = 2^{\operatorname{card}(A)}$.

**Acknowledgements.** I am grateful to H. Draškovičová and M. Kolibiar for informing me of the problem and for their advice.

**References**


Department of Algebra, Faculty of Sciences, University of Komenský, 816 31 Bratislava, Czechoslovakia