A TERNARY FUNCTION FOR
DISTRIBUTIVITY AND PERMUTABILITY
OF AN EQUIVALENCE LATTICE

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Abstract. The main result of the paper is

Theorem 1. Let $A$ be a countable set and $L$ be a complete sublattice of the
equivalence lattice on $A$. The following are equivalent

(i) $L$ is a distributive lattice of permutable equivalence relations.
(ii) There is an algebra with congruence lattice $L$ among the fundamental
operations of which is a ternary function $f$ with the property

$$f(a, b, b) = f(a, b, a) = f(b, b, a) = a$$

for all $a, b \in A$.

This theorem is a contribution to the concrete representation problem for
congruence lattices. Other results related to this problem can be found in [2].
We always assume that a complete sublattice of a complete lattice $U$ has the
same extremal elements as $U$. Suppose $\equiv$ is an equivalence relation on the set
$A$ and $g$ is a function from some subset $X$ of $A^3$ into $A$. We write
$(a, b, c) \equiv (d, e, f)$ for $a \equiv d$, $b \equiv e$, $c \equiv f$; $g$ is compatible with $\equiv$ provided
$g(a, b, c) \equiv g(d, e, f)$ whenever $(a, b, c) \in X$, $(d, e, f) \in X$ and
$(a, b, c) \equiv (d, e, f)$; $g$ is compatible with a set $L$ of congruence relations if it is
compatible with each member of $L$. Analogous definitions are used for unary
functions. In the proof of Theorem 1 we use:

Theorem 2. Let $L$ be a complete distributive lattice of permutable equivalence
relations on the countable set $A$. There is a function $f$ from $A^3$ into $A$ which is
compatible with $L$ and for which (1) holds for all $a, b \in A$.

An analogous theorem, for $A$ arbitrary and $L$ finite, was proved in [3] ($L$
was considered as a lattice of congruences of some algebra; however this fact
was not substantially used). At the Colloquium on Universal Algebra in
Oberwolfach, July 1973, A. F. Pixley asked whether the finiteness of $L$ can be
omitted in his theorem. Our theorems give a partial answer to his question.

Proof of Theorem 2. For the sake of convenience we suppose that $A$ is
countably infinite. The finite case may be obtained by halting our
construction of $f$ below at the appropriate place, but this case was fully
established in [3] (cf. Lemma 3.1 where the assumption that $L$ is the
congruence lattice of some algebra is extraneous). We may also suppose
\( A^2 \subseteq L \). Let
\[
D = \{(a, b, c) \in A^3 : a, b, c \text{ are pairwise different}\},
\]
\( F = A^3 - D \). Now \( D \) is countably infinite, so we impose on it the order type of the natural numbers: \( t_0 < t_1 < t_2 \ldots \). Define
\[
S(a, b, c) = \{(d, e, f) : \{d, e, f\} \subseteq \{a, b, c\}\}
\]
and let
\[
g(a, b, c) = \begin{cases} 
  a & \text{if } b = c \\
  c & \text{otherwise}
\end{cases} \quad \text{for } (a, b, c) \in F.
\]
It is easy to verify that \( g \) is compatible with all equivalence relations on \( A \).

**Lemma 1.** If the domain of \( h_k \) consists of all \( t < t_k \) with \( t \in D \) and \( h_k \cup g \) is compatible with \( L \), then \( h_k \) can be extended to \( h_{k+1} \) with the domain \( \{t_i : 0 < i < k\} \) so that \( h_{k+1} \cup g \) is compatible with \( L \).

**Proof of Lemma 1.** Let \( f_k = h_k \cup g \) and \( E_k = \{t_i : 0 < i < k\} \cup S(t_k) \). \( E_k \) is finite and \( E_k \) is a subset of the domain of \( f_k \). For each \( t \in E_k \) let \( \tilde{t} \) be the least equivalence relation in \( L \) with \( t \not\equiv t_k \). Evidently \( t \not\equiv \tilde{t}, s \) for all \( t, s \in E_k \). Since \( \tilde{t}, \tilde{t}_2, \tilde{t}_3 \in L \), we have \( \tilde{t}, \tilde{t}_2, \tilde{t}_3 \in L \), so by the compatibility of \( f_k \) we obtain \( f_k(t) \not\equiv f_k(s) \). Consequently, by the Chinese Remainder Theorem (see Pixley [3] or Grätzer [1, p. 211, Exercise 68]) we conclude that there is \( d \in A \) such that \( f_k(t) \equiv d, d \) for all \( t \in E_k \). Let \( h_{k+1}(t_k) \) be such a \( d \). To see that \( f_{k+1} \) \((= h_{k+1} \cup g) \) is compatible with \( L \) suppose that \( t \in E_k \cup F \) and \( t \not\equiv t_k \) for some \( \tilde{t} \in L \). If \( t \in E_k \) we have \( \tilde{t}_2, s \subseteq \tilde{t}, t_k \). Then \( f_{k+1}(t) = f_k(t) \not\equiv f_{k+1}(t_k) = f_{k+1}(t_k) \), and so \( f_{k+1}(t) \not\equiv f_{k+1}(t_k) \). If \( t \in F \) we can suppose \( t = (a, a, b) \); the other two cases are similar. Let \( t_k = (x, y, z) \). Then \( x \not\equiv a, y, z \not\equiv b \). So \( x \not\equiv y \) and we obtain \((x, x, z) \not\equiv t_k \). But \((x, x, z) \in S(t_k) \subseteq E_k \). Hence
\[
f_{k+1}(t_k) \not\equiv f_k(x, x, z) = z \not\equiv b = f_{k+1}(a, a, b).
\]
This completes the proof of Lemma 1.

By well ordering \( A \) we can define \( f \) by the following recursion using Lemma 1 to do the crucial step.
\[
h_0 \text{ is the empty function;}
\]
\[
d_k \text{ is the least } e \in A \text{ such that } h_k \cup \{(t_k, e)\} \cup g \text{ is compatible with } L;
\]
\[
h_{k+1} = h_k \cup \{(t_k, d_k)\}.
\]
Then let \( f = g \cup \bigcup_{k=0}^\infty h_k \). Observe that (1) holds since \( g \subseteq f \). Let \( t, s \in A^3 \) and suppose \( t \not\equiv s \) for some \( \tilde{t} \in L \). There is a \( k \) such that \( t \) and \( s \) are both in the domain of \( g \cup h_k \). Since \( g \cup h_k \) is compatible with \( L \) we conclude
\[
f(t) = (g \cup h_k)(t) \not\equiv (g \cup h_k)(s) = f(s).
\]
In this way Theorem 2 is established.

**Proof of Theorem 1.** It suffices to prove that (i) implies (ii); the converse was proved in [3, p. 183]. If (i) holds, then by Theorem 2 there is a function \( f \)
from $A^3$ into $A$ which is compatible with $L$ and which satisfies (1) for all $x, z \in A$; it will be one of the fundamental operations of the algebra which we construct. The remainder fundamental operations of it will exclude the equivalence relations not belonging to $L$.

**Lemma 2.** If $\eta \not\in L$ is an equivalence relation on $A$ then there is a unary operation $g$ on $A$ which is compatible with $L$ and not compatible with $\eta$.

**Proof of Lemma 2.** Let $\vartheta_{(x,y)}$ be the least element $\xi$ of the lattice $L$ such that $x \leq \xi y$. Then obviously $\eta \leq \bigvee_{(a,b)\in\eta} \vartheta_{(a,b)}$. However, $\eta \not\in L$ and thus the equality does not hold. Therefore there is $(a_0, a_1) \in \eta$ such that $\vartheta_{(a_0,a_1)} \not\in \eta$; hence there are $c_0, c_1$ such that $\vartheta_{(c_0,c_1)} \vartheta_{(a_0,a_1)} c_1$. Take these $a_0, a_1, c_0, c_1$ and arbitrary $\xi \in L$. Then $a_0 \not\leq a_1$ implies $\xi \not\leq \vartheta_{(a_0,a_1)}$ and, hence, $c_0 \not\leq c_1$. Thus the unary partial function $g_2 = \{(a_0, c_0), (a_1, c_1)\}$ is compatible with $L$; it obviously is not compatible with $\eta$. (Up to this moment we have used neither countability of $A$ nor (i).)

Impose on $A - \{a_0, a_1\}$ the order type of natural numbers $a_2 < a_3 < a_4 < \ldots$. Suppose that $k > 2$ and that $g_k = \{(a_0, c_0), (a_1, c_1), \ldots, (a_{k-1}, c_{k-1})\}$ is constructed, $g_k$ is compatible with $L$. Let $\vartheta_i (0 < i < k)$ be the least element of $L$ such that $a_i \vartheta_i a_k$. In the same way as in Lemma 1 we can find the least $j$ such that $c_i \vartheta_j a_j$ for all $i < k$, and show that $g_{k+1} = g_k \cup \{(a_k, c_k)\}$, where $c_k = a_j$, is compatible with $L$. Then $g = \bigcup_{k=2}^{\infty} g_k$ is also compatible with $L$. Since $a_0 \vartheta a_1$, $\vartheta g(a_0) \eta g(a_1)$, the function $g$ is not compatible with $\eta$. Lemma 2 is established.

For every equivalence relation $\vartheta$ on $A$, $\vartheta \not\in L$, let $g_{\vartheta}$ be a unary function compatible with $L$ and not compatible with $\vartheta$. Let the set of fundamental operations of an algebra $\mathfrak{A}$ consist of $f$ and all $g_{\vartheta}$; their ordering is not important. Then $L$ is the congruence lattice of $\mathfrak{A}$. Q.E.D.

**Remarks.** 1. After $g_2$ was constructed in the proof of Lemma 2 we have not used $\eta$ in the construction of $g$. Hence we may ask that the algebra in (ii) has countable signature.

2. Using the first part of the proof of Lemma 2 we can show: If $A$ is an infinite set, $B$ is the set of all complete sublattices of the equivalence lattice on $A$, then $\text{card}(B) = 2^{\text{card}(A)}$.

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**References**


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